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THE EXTENDING OF DARBOUX FUNCTIONS WITH FINITE VARIATION

Abstract

In this paper we investigate the problem of extending Darboux functions with finite varation in such a way that the sets of points of quasi-continuity are maintained.

Paper [1] was devoted, among other things, to the investigation of the possibility of extending Darboux functions defined on closed and convex subsets of the plane, taking their values in $\mathbb{R}^2$, in such a way that the original functions and their extensions should possess the same sets of points of continuity.

In the present paper, the problem of extending Darboux functions with finite variation is studied. At the same time, we consider the situation where the sets of points of quasi-continuity of the extension and of the original function are equal. However, quasi-continuity differs from continuity in that it may be 'realized from different sides'. Thereby, in Theorem 1 we demand not only that the sets of points of quasi-continuity of suitable functions be equal, but also that, for points of quasi-continuity of the extended function, the image of the open set non-disjoint from the complement of the domain of the original transformation should not be contained in a 'close neighbourhood' of the images of these points (the consideration of the notation $\tilde{Q}_{f^*}(\mathcal{K})$).

At the same time, it seems interesting to ask not only about the possibility of extending a given function to the whole plane, but also about the existence of extensions to some sets of the plane, with that we sought for as weak assumptions as possible, concerning the sets we extend the original transformations to. Let us note here that in this situation one cannot use the method of extending the transformation to the whole plane in order to restrict next the extended function to a fixed subset. As a result of this operation, the terminal transformation could possess many more points of quasi-continuity than the original function although the 'intermediate mapping' (mapping $\mathbb{R}^2$ into $\mathbb{R}^2$) would satisfy the assertion of Theorem 1.

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Throughout the paper, we adopt the standard symbols, definitions and notations. The letters $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ denote: the sets of all positive integers, rational numbers and real numbers, respectively. Let $\mathcal{Z}$ be an arbitrary set and let $\mathcal{R}$ stand for an equivalence relation; then we denote the set of all abstract classes of the relation $\mathcal{R}$ by $\mathcal{Z}/\mathcal{R}$. Let $\mathcal{X}$ stand for a metric space; then we denote an arc in $\mathcal{X}$ with endpoints a, b by $L(a, b)$. If a, b are points belonging to the arc $\mathcal{L} \subset \mathcal{X}$, then $L_{\mathcal{L}}(a, b)$ denotes the only arc included in $\mathcal{L}$ with endpoints a, b. The two-dimensional Lebesgue measure of the set $A \subset \mathbb{R}^2$ is denoted by $m_2(A)$. Let $x, y \in \mathbb{R}^2$. The symbols H_x, H_x^y mean: a half-line in $\mathbb{R}^2$ with initial point x, and a half-line in $\mathbb{R}^2$ with initial point x which includes a point y, respectively.

Let p, q be two half-lines with the same initial point. It is known that the figure $p \cup q$ cuts the plane $\mathbb{R}^2$ into two domains K_1^0, K_2^0. At least one of these domains is convex. Each *convex* set $K_1^0 \cup p \cup q$, $K_2^0 \cup p \cup q$ will be called an angle determined by half-lines p and q and denoted by $\angle(p, q)$. The arc measure of the angle $\angle(p, q)$ is denoted by the symbol $m\angle(p, q)$.

Let $\mathcal{Y}$ be a subspace of the space $\mathbb{R}^2$, let ξ denote a real nonnegative number and let $\emptyset \neq \mathcal{K} \subset \mathcal{Y}$. Then the symbol $\mathcal{K}_\xi^{\mathcal{Y}}$ stands for the set $\{y \in \mathcal{Y} : d(\mathcal{K}, y) = \xi\}$. We shall write $\mathcal{K}_\xi$ instead of $\mathcal{K}_\xi^{\mathbb{R}^2}$. Let $B \subset \mathcal{Y}$. The set $\{\xi \in \mathbb{R} : \xi > 0$ and $B \cap \mathcal{K}_\xi^{\mathcal{Y}} \neq \emptyset\}$ will be denoted by $B_{\mathcal{K}}^{\mathcal{Y}}$, or by $B_{\mathcal{K}}$ when it is clear in what space our considerations take place.

In the latter part of the paper we shall make use of spaces forming a wider class than locally connected ones. These spaces will be called stratiformly locally connected.

Definition 1 *We say that the space $\mathcal{Y}$ is stratiformly locally connected with respect to $\mathcal{K} \subset \mathcal{Y}$ ($\mathcal{K} \neq \emptyset$) if there exists a base B of this space such that $V_{\mathcal{K}}$ is a connected set for any $V \in B$.*

Definition 2 *Let $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$, $\mathcal{Y}$ are arbitrary topological spaces. We say that f is a Darboux function if the image of each arc $\mathcal{L} \subset \mathcal{X}$ is a connected set.*

Let $f : \mathcal{Y} \to \mathbb{R}^2$. The symbols C_f and Q_f denote the sets of all continuity points and the quasi-continuity points, respectively.

Let F be a closed subset in $\mathcal{Y}$. We write $x_0 \in \tilde{Q}(F)$ when $x_0 \in Q_f$ and there exists a neighbourhood V of a point $f(x_0)$ in $\mathbb{R}^2$ such that, for each neighbourhood W of the point x_0, open in $\mathcal{Y}$, the following condition: $\mathrm{Int}_{\mathcal{Y}}(W \cap f^{-1}(V)) \subset F$ is satisfied.

Let $\{A_s\}_{s \in S}$ be a cover of the space $\mathcal{Y}$ and let $\{f_s\}_{s \in S}$ be a family of agree transformations, where $f_s : A_s \to \mathbb{R}^2$ for $s \in S$. Then the symbol $\underset{s \in S}{\nabla} f_s$

denotes a combination of transformations $\{f_s\}_{s\in S}$. When $S = \{1, 2, \ldots, k\}$, the notation $f_1 \triangledown f_2 \triangledown \cdots \triangledown f_k$ is also used.

We shall need the following lemma (cf. [1], Lemma 2).

Lemma 1 *Let $\mathcal{K} \subset \mathbb{R}^2$ ($\mathcal{K} \neq \emptyset$) be a closed convex set and let $p \in \operatorname{Int} \mathcal{K}$. Then if there exists non-negative real number ξ_0 such that $H_p \cap \mathcal{K}_{\xi_0} \neq \emptyset$, then, for any $\xi \geq 0$, the intersection $H_p \cap \mathcal{K}_\xi$ is a one-element set.*

The formulation of the main theorem of this paper will be preceded by the following lemma

Lemma 2 *Let $\mathcal{K} \neq \mathbb{R}^2$ be a convex and closed subset of the plane $\mathbb{R}^2$ such that $\operatorname{Int} \mathcal{K} \neq \emptyset$. Then, for each element $p \in \operatorname{Int} \mathcal{K}$, each real number $\xi > 0$ and any $x \in \mathcal{K}_\xi$, there exist arcs $\mathcal{L} = L(x, a)$, $\mathcal{L}' = L(x, b)$ such that $\mathcal{L}, \mathcal{L}' \subset \mathcal{K}_\xi$ and the sets $\mathcal{L}\backslash\{x\}$, $\mathcal{L}'\backslash\{x\}$ are contained in distinct open half-planes determined by the line the points p and x belong to.*

PROOF. Let $p \in \operatorname{Int} \mathcal{K}$, $\xi > 0$ and $x \in \mathcal{K}_\xi$. Consider the ball $K(x, \xi)$. The closedness of the set $\mathcal{K}$ implies the existence of $k_x \in \mathcal{K}$ such that $d(k_x, x) = \xi$. Let l be the line tangent to the sphere $S(x, \xi)$ at the point k_x.

Note that

$$p \notin P(l, K(x, \xi)) \tag{1}$$

where $P(l, K(x, \xi))$ stands for a closed half-plane with edge l, containing the ball $K(x, \xi)$.

We shall show that

$$\mathcal{K} \subset P(l, p) \tag{2}$$

where $P(l, p)$ denotes a closed half-plane with edge l, containing the point p. Indeed, let us suppose it is not so. Then there exists

$$e \in \mathcal{K}\backslash P(l, p). \tag{3}$$

Relations (1) and (3) allow us to deduce that $e \in P(l, K(x, \xi))\backslash l$. Since l is a line tangent to $S(x, \xi)$ at the point k_x, there exists an element e' of the non-degenerate segment $[k_x, e]$, such that

$$e' \in K(x, \xi). \tag{4}$$

In view of the convexity of $\mathcal{K}$, from (3) and the condition $k_x \in \mathcal{K}$ it follows that $e' \in \mathcal{K}$. The last condition and (4) imply that $x \notin \mathcal{K}_\xi$, which leads to a contradiction. Condition (2) has been shown.

In view of (1) and (2), it is not hard to observe that

$$\mathcal{K}_\xi \subset P(m,p) \tag{5}$$

where m is a line parallel to l such that $x \in m$; moreover, $P(m,p)$ stands for a closed half-plane with edge m, containing the point p.

Let $z \in K(x,\xi)\backslash H_p^x$ be a point such that $\angle(H_p^x, H_p^z)$ has a negative orientation. Denote $\alpha = m\angle(H_p^x, H_p^z)$.

Denote a function $t : [0,\alpha] \to \mathbb{R}^2$ in the following way: for any $y \in [0,\alpha]$, let $H_{p,y}$ denote a half-line with the initial point at p, such that

$$\angle(H_p^x, H_{p,y}) \subset \angle(H_p^x, H_p^z) \qquad \text{and} \qquad m\angle(H_p^x, H_{p,y}) = y.$$

Let $\{s_y\} = H_{p,y} \cap \mathcal{K}_\xi$. In virtue of Lemma 1, the point s_y exists and is determined uniquely. Let $t(y) = s_y$.

The function t is, of course, injective. We shall show the continuity of this transformation.

Let $x_0 \in [0,\alpha]$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of the interval $[0,\alpha]$, such that $x_n \to x_0$. In view of the definition of $t(x_n)$ for $n = 1,2,\ldots$ and (5) it follows that

$$t(x_n) \in T \qquad \text{for} \qquad n = 1,2,\ldots \tag{6}$$

where T is a triangle determined by the half-lines H_p^z, H_p^x and the line m.

Let $\{t(x_{k_n})\}$ be any subsequence of $\{t(x_n)\}$. In virtue of (6), we can choose a subsequence $\{t(x_{l_{k_n}})\}$ of $\{t(x_{k_n})\}$ converging to some

$$q \in \angle(H_p^x, H_p^z). \tag{7}$$

We shall demonstrate that

$$q \in H_{p,x_0} \tag{8}$$

where H_{p,x_0} is a half-line with the initial point p, such that $H_{p,x_0} \subset \angle(H_p^x, H_p^z)$ and $m\angle(H_p^x, H_{p,x_0}) = x_0$. Suppose it is not so. There are two cases possible.

1^0 Let $x_0 \in (0,\alpha)$. Then there exist distinct half-lines L_1, L_2 with the initial point p, such that

$$L_1, L_2 \subset \angle(H_p^x, H_p^z), \qquad H_{p,x_0} \subset \angle(L_1, L_2) \qquad \text{and} \qquad q \notin \angle(L_1, L_2)$$

and

$$m\angle(L_1, H_p^x) = a_1 \in (0, x_0) \qquad \text{and} \qquad m\angle(L_2, H_p^x) = b_1 \in (x_0, \alpha).$$

Hence it appears that $L_1 = H_{p,a_1}$ and $L_2 = H_{p,b_1}$ and $x_0 \in (a_1, b_1)$. The way of choosing the lines L_1, L_2 implies that $q \in \mathbb{R}^2 \setminus \angle(L_1, L_2)$. In view of the way of choice of the sequence $\{t(x_{l_{k_n}})\}$ and the definition of the mapping t, $H_{p,x_{l_{k_n}}} \subset \mathbb{R}^2 \setminus \angle(L_1, L_2)$ for any $n \geq n_0$ where n_0 is some positive integer. Consequently, $x_{l_{k_n}} \notin [a_1, b_1]$ for $n \geq n_0$. This contradicts the convergence of the sequence $\{x_n\}$ to the point x_0.

2^0 Let $x_0 = 0$ or $x_0 = \alpha$. Assume, without loss of generality, that $x_0 = 0$. Then there exists a half-line L with the initial point p, such that

$$L \subset \angle(H_p^x, H_p^z), \qquad q \notin \angle(H_p^x, L) \qquad \text{and} \qquad m\angle(H_p^x, L) = b' \in (0, \alpha).$$

Hence it appears that $L = H_{p,b'}$. In view of the definition of the line L and (7), it follows that $q \in \angle(H_p^x, H_p^z) \setminus \angle(H_p^x, L)$. Reasoning similarly as in 1^0, we infer that $x_{l_{k_n}} \notin [0, b']$ for $n \in \mathbb{N}$ sufficiently large. This, however, contradicts the convergence of the sequence $\{x_n\}$ to the point x_0.

Condition (8) has thus been shown. In virtue of the closedness of $\mathcal{K}_\xi$, we conclude that $q \in \mathcal{K}_\xi$. This means that $q \in \mathcal{K}_\xi \cap H_{p,x_0}$. In view of Lemma 1 and the definition of the mapping t, $q = t(x_0)$. The continuity of t has finally been proved.

Thereby, we have demonstrated that t is a homeomorphism of the segment $[0, \alpha]$ onto the set $\mathcal{L} = t([0, \alpha])$. The arc $\mathcal{L}$ satisfies, as is easily seen, the conditions of the assertion of the lemma.

The construction of the other arc ($\mathcal{L}'$) runs analogously.

Theorem 1 *Let $\mathcal{K}$ be a closed convex subset of the plane $\mathbb{R}^2$ which is not a boundary set and let $f : \mathcal{K} \to \mathbb{R}^2$ be a Darboux function with finite variation. Then, for each connected subspace $\mathcal{X}$ of the space $\mathbb{R}^2$, stratiformly locally connected with respect to $\mathcal{K}$, containing $\mathcal{K}$, there exists a Darboux function $f^* : \mathcal{X} \to \mathbb{R}^2$ with finite variation, being an extension of f to $\mathcal{X}$, such that*

$$Q_f = Q_{f^*} \subset \tilde{Q}_{f^*}(\mathcal{K}).$$

PROOF. In virtue of the assumptions, there exists a point $p \in \operatorname{Int} \mathcal{K}$. Let $\mathcal{S}$ be the union of all lines parallel to the co-ordinate axes and intersecting these axes at points with rational co-ordinates. Then

$$m_2(\mathcal{S}) = 0 \qquad \text{and} \qquad \bar{S} = \mathbb{R}^2. \tag{1}$$

To make the further notation easier, let $\Theta = \mathcal{X}_\mathcal{K}$. It is easy to notice that if $\Theta = \emptyset$, then $\mathcal{K} = \mathcal{X}$ and it suffices to adopt $f^* = f$. So, assume that $\Theta \neq \emptyset$ (hence it appears that $\mathcal{X} \setminus \mathcal{K} \neq \emptyset$). Note that

$$\mathcal{X} = \mathcal{K} \cup \bigcup_{\xi \in \Theta} \mathcal{K}_\xi^\mathcal{X}. \tag{2}$$

Let

$$\begin{array}{l}\text{the symbol } \mathcal{A} \text{ denote the set of all those elements } x \text{ of the} \\ \text{space } \mathcal{X} \text{ for which there exists a neighbourhood } V_x \text{ of the} \\ \text{point } x \text{ in the space } \mathcal{X}, \text{ such that } V_x \subset \mathcal{K}^{\mathcal{X}}_{\xi_x} \\ \text{where } \xi_x \text{ is a positive number such that } x \in \mathcal{K}^{\mathcal{X}}_{\xi_x},\end{array} \tag{3}$$

Th

and, to simplify the following notation, let $\Theta' = \mathcal{A}^{\mathcal{X}}_{\mathcal{K}}$.
We shall demonstrate that

$$\mathcal{K}^{\mathcal{X}}_{\xi} \cap \mathcal{A} \text{ is an open subset of the space } \mathcal{X} \text{ for any } \xi \in \Theta'. \tag{4}$$

Indeed, let $\xi \in \Theta'$. Then there exists $x \in \mathcal{K}^{\mathcal{X}}_{\xi} \cap \mathcal{A}$. By virtue of (3), it follows that there exists a neighbourhood V of the point x, open in $\mathcal{X}$, such that $V \subset \mathcal{K}^{\mathcal{X}}_{\xi}$. The last condition and the openness of V prove the veracity of the inclusion $V \subset \mathcal{A}$. Condition (4) has been demonstrated.

At present, we shall define a function $g : \mathcal{A} \xrightarrow{\text{onto}} \mathcal{S}$. With that end in view, for each $\xi \in \Theta'$, define a function $g_\xi : \mathcal{K}^{\mathcal{X}}_{\xi} \cap \mathcal{A} \xrightarrow{\text{onto}} \mathcal{S}$. Let, for any $\xi \in \Theta'$, $\mathcal{M}_\xi$ denote the family of all components of the set $\mathcal{K}^{\mathcal{X}}_{\xi} \cap \mathcal{A}$ and let, for any $C \in \mathcal{M}_\xi$, the symbol x_C denote some element of the component C (in order to avoid the necessity of repeating analogous explanations, the distinguished element of a given component will always be denoted by the symbol x with the index being the symbol of this component).

Fix $\xi \in \Theta'$ and $C \in \mathcal{M}_\xi$. Let $\alpha_z = m\angle(H^{x_C}_p, H^z_p)$ for any $z \in C$ and let $N = \{\alpha_z : z \in C\}$. Note that

$$\operatorname{Int}_{\mathbb{R}} N \neq \emptyset. \tag{5}$$

Indeed, since $x_C \subset \mathcal{K}^{\mathcal{X}}_{\xi} \cap \mathcal{A}$, therefore, in view of (4), we may deduce that

$$\overline{K(x_C, \varepsilon)} \cap \mathcal{X} \subset \mathcal{K}^{\mathcal{X}}_{\xi} \qquad \text{for some } \ \varepsilon > 0, \tag{6}$$

where $K(x_C, \varepsilon)$ is a ball in $\mathbb{R}^2$ and the closure operation is considered in $\mathbb{R}^2$, too. Let $\mathcal{L} = L(x_C, a) \subset \mathcal{K}_\xi$ and $\mathcal{L}' = L(x_C, b) \subset \mathcal{K}^{\mathcal{X}}_{\xi}$ be the arcs, described in Lemma 2, for the point x_C and let $\{p_1, p_2\} = (\operatorname{Fr} K(x_C, \varepsilon)) \cap H^{x_C}_p$. We may assume that $p_1 \in (p, p_2)$. In view of Lemma 1, $p_1, p_2 \notin \mathcal{K}_\xi$. By virtue of the continuity of the function $p \mapsto d(\mathcal{K}, p)$, we may infer that there exist numbers $\delta_1, \delta_2 > 0$ satisfying the condition

$$K(p_i, \delta_i) \cap \mathcal{K}_\xi = \emptyset \qquad \text{for} \qquad i = 1, 2. \tag{7}$$

Let $\Gamma_a = \angle(H_p^{x_C}, H_p^a)$, $\Gamma_b = \angle(H_p^{x_C}, H_p^b)$ and $\Gamma = \Gamma_a \cup \Gamma_b$. Moreover, let L be a line containing $H_p^{x_C}$. The definition of the angle and the way of defining the arcs $\mathcal{L}$ and $\mathcal{L}'$ imply that

$$(L \backslash H_p^{x_C}) \cap \Gamma = \emptyset \qquad \text{and} \qquad H_p^{x_C} \subset \Gamma. \tag{8}$$

We shall now show that

$$\begin{aligned} &\text{there exists } q_1 \text{ such that } L_{\mathcal{L}}(x_C, q_1) \subset \mathcal{A} \\ &\text{or there exists } q_2 \text{ such that } L_{\mathcal{L}'}(x_C, q_2) \subset \mathcal{A}. \end{aligned} \tag{9}$$

Suppose to the contrary that condition (9) is not satisfied. This means that there exist sequences $\{q_n^1\}_{n\in\mathbb{N}} \subset \mathcal{L}\backslash(\mathcal{A} \cup \{x_C\})$ and $\{q_n^2\}_{n\in\mathbb{N}} \subset \mathcal{L}'\backslash(\mathcal{A} \cup \{x_C\})$ such that

$$\lim_{n\to\infty} q_n^1 = \lim_{n\to\infty} q_n^2 = x_C.$$

So, let n_0 be a positive integer such that

$$\begin{aligned} &\Gamma_1 = \angle(H_p^{x_C}, H_p^{q_{n_0}^1}) \cup \angle(H_p^{x_C}, H_p^{q_{n_0}^2}) \subset \Gamma, \\ &\Gamma_1 \cap \operatorname{Fr} K(x_C, \varepsilon) \subset K(p_1, \delta_1) \cup K(p_2, \delta_2) \end{aligned} \tag{10}$$

and

$$L_{\mathcal{L}}(x_C, q_{n_0}^1) \cup L_{\mathcal{L}'}(x_C, q_{n_0}^2) \subset K(x_C, \varepsilon). \tag{11}$$

Denote by the symbol P_i segments of $\overline{K(x_C, \varepsilon)} \cap H_p^{q_{n_0}^1}$ for $i = 1, 2$. Then, by virtue of (6), $P_i \cap \mathcal{X} \subset \mathcal{K}_\xi^{\mathcal{X}}$ for $i = 1, 2$. By Lemma 1, $P_i \cap \mathcal{K}_\xi = \{q_{n_0}^i\}$ for $i = 1, 2$. Let i be a fixed element of the set $\{1, 2\}$. If $q_{n_0}^i \in \mathcal{X}$, then, in virtue of (6) and the condition $q_{n_0}^i \in K(x_C, \varepsilon)$ following from (11), we would have, by (3), the relation $q_{n_0}^i \in \mathcal{A}$. This leads, in view of the way of defining the element $q_{n_0}^i$, to a contradiction. This means that $q_{n_0}^i \notin \mathcal{X}$ for $i = 1, 2$. In consequence, $P_i \cap \mathcal{X} = \emptyset$. In view of (7) and (10), $\Gamma_1 \cap \operatorname{Fr}(K(x_C, \varepsilon)) \cap \mathcal{K}_\xi = \emptyset$. Relationship (6) allows us to deduce that $\Gamma_1 \cap \operatorname{Fr}(K(x_C, \varepsilon)) \cap \mathcal{X} = \emptyset$. The reasoning carried out leads to the conclusion that if $\mathcal{T} = P_1 \cup P_2 \cup (\Gamma_1 \cap \operatorname{Fr}(K(x_C, \varepsilon)))$, then $\mathcal{T} \cap \mathcal{X} = \emptyset$. Since $\mathcal{T}$ cuts $\mathbb{R}^2$ into two separated sets, each of them containing elements of the connected set $\mathcal{X}$, therefore $\mathcal{T} \cap \mathcal{X} \neq \emptyset$. The contradiction obtained completes the proof of (9).

In view of (9) and the fact that $\mathcal{L} \cup \mathcal{L}' \subset \mathcal{K}_\xi$, we may deduce that the component C contains some arcs $L_{\mathcal{L}}(x_C, q_1)$ or $L_{\mathcal{L}'}(x_C, q_2)$. Assume, with no loss of generality, that $L_{\mathcal{L}}(x_C, q_1) \subset C$. Let $\alpha_0 = m\angle(H_p^{x_C}, q_1)$; then $(0, \alpha) \subset N$.

Indeed, let $\psi \in (0, \alpha)$ and let H_ψ, be a half-line with the initial point p, such that $\angle(H_p^{x_C}, H_\psi) \subset \angle(H_p^{x_C}, H_p^{q_1}) \subset \Gamma$ and $m\angle(H_p^{x_C}, H_\psi) = \psi$. It is sufficient to show that $H_{\psi} \cap L_{\mathcal{L}}(x_C, q_1) \neq \emptyset$. Suppose that $H_{\psi} \cap L_{\mathcal{L}}(x_C, q_1) = \emptyset$. Then the figure $(L \backslash H_p^{x_C}) \cup H_{\psi}$ cuts $\mathbb{R}^2$ into separated sets such that x_C belongs to one of them, and q_1 to the other, which leads to a contradiction.

The proof of condition (5) is finished.

In the set N we shall define an equivalence relation $\mathcal{R}_1$ in the following manner:

$a\mathcal{R}_1 b \Leftrightarrow a - b$ is a rational number (for any $a, b \in N$).

Let $N^* = N/\mathcal{R}_1$. By virtue of (5),

$$\text{the cardinality of the set } N^* \text{ equals the continuum.} \tag{12}$$

By (12), it follows that there exists a bijection $\mu : N^* \to \mathcal{S}$. Let a function $g_\xi^C : C \xrightarrow{\text{onto}} \mathcal{S}$ be defined as follows:

$$g_\xi^C(c) = \mu([m\angle(H_p^{x_C}, H_p^c)]_{\mathcal{R}_1}) \qquad \text{for any } c \in C,$$

where the symbol $[\ ]_{\mathcal{R}_1}$ stands for the abstract class of the relation $\mathcal{R}_1$.

Put

$$g_\xi = \mathop{\nabla}_{C \in \mathcal{M}_\xi} g_\xi^C : \mathcal{K}_\xi^{\mathcal{X}} \cap \mathcal{A} \xrightarrow{\text{onto}} \mathcal{S}$$

and let

$$g = \mathop{\nabla}_{\xi \in \Theta'} g_\xi : \mathcal{A} \xrightarrow{\text{onto}} \mathcal{S}.$$

At present, we shall define a function $h : \mathcal{X} \backslash (\mathcal{A} \cup \mathcal{K}) \xrightarrow{\text{onto}} \mathcal{S}$. For the purpose, we first define an equivalence relation $\mathcal{R}_2$ by the following formula:

$\alpha_1 \mathcal{R}_2 \alpha_2 \Leftrightarrow \alpha_1 - \alpha_2$ is a rational number (for any $\alpha_1, \alpha_2 \in \Theta$).

Let $\Theta/\mathcal{R}_2 = \hat{N}$. Since $\Theta \neq \emptyset$, and $\mathcal{X}$ is a connected space, therefore

$$\text{the cardinality of the set } \hat{N} \text{ equals the continuum.} \tag{13}$$

Condition (13) implies the existence of a bijection $\nu : \hat{N} \to \mathcal{S}$. For any $x \in \mathcal{X} \backslash (\mathcal{A} \cup \mathcal{K})$, let

$$h(x) = \nu([\alpha_x]_{\mathcal{R}_2})$$

where the symbol $[\ ]_{\mathcal{R}_2}$ denotes the abstract class of the relation $\mathcal{R}_2$ and α_x - a positive real number such that $x \in \mathcal{K}_{\alpha_x}^{\mathcal{X}}$.

Let $f^* = f \bigtriangledown h \bigtriangledown g : \mathcal{X} \to \mathbb{R}^2$. We shall demonstrate that f^* is the sought-for function. To this end, we shall first show that f^* has the Darboux property.

Let $\mathcal{L}_1 \subset \mathcal{X}$ be an arc. It should be shown that $f^*(\mathcal{L}_1)$ is a connected set. The following cases are possible:

I. $\mathcal{L}_1 \cap \mathcal{A} \neq \emptyset$. Then there exists $x_0 \in \mathcal{L}_1 \cap \mathcal{A}$. Let ξ_0 be a number such that $x_0 \in \mathcal{K}^{\mathcal{X}}_{\xi_0} \cap \mathcal{A}$. By virtue of (4),

$$K(x_0, \eta) \cap \mathcal{X} \subset \mathcal{K}^{\mathcal{X}}_{\xi_0} \cap \mathcal{A} \qquad \text{for some } \eta > 0. \tag{14}$$

Let $z_0 \in \mathcal{L}_1 \backslash \{x_0\}$ be a point such that $\tilde{\mathcal{L}} = L_{\mathcal{L}_1}(x_0, z_0) \subset K(x_0, \eta) \cap \mathcal{X}$ and let $C_{x_0} \in \mathcal{M}_{\xi_0}$ be a component of the set $\mathcal{K}^{\mathcal{X}}_{\xi_0} \cap \mathcal{A}$, containing x_0. Since, by (14), $\tilde{\mathcal{L}} \subset \mathcal{K}^{\mathcal{X}}_{\xi_0} \cap \mathcal{A}$, therefore $\tilde{\mathcal{L}} \subset C_{x_0}$. For each $l \in \tilde{\mathcal{L}}$, denote

$$\alpha_1 = m\angle(H_p^{x_{C_{x_0}}}, H_p^l).$$

Besides, let us adopt $N_1 = \{\alpha_1 : l \in \tilde{\mathcal{L}}\}$.

Note first that N_1 is not a one-element set.

So, let β_1, β_2 ($\beta_1 < \beta_2$) be elements of the set N_1. We shall first show the veracity of the inclusion

$$[\beta_1, \beta_2] \subset N_1. \tag{15}$$

Indeed, for let us assume that this inclusion does not hold, that is,

$$\text{there exists } \beta \in (\beta_1, \beta_2) \backslash N_1. \tag{16}$$

Let L_1, L_2 be distinct half-lines with the initial point p, such that

$$m\angle(H_p^{x_{C_{x_0}}}, L_1) = m\angle(H_p^{x_{C_{x_0}}}, L_2) = \beta.$$

Then (16) and the definition of the set N_1 imply the equality

$$(L_1 \cup L_2) \cap \tilde{\mathcal{L}} = \emptyset. \tag{17}$$

Of course, $L_1 \cup L_2$ cuts the plane into two disjoint open sets U_1, U_2. Without loss of generality it may be assumed that $x_{C_{x_0}} \in U_1$. Since $\beta_1 \in N$ and $\beta_1 < \beta$, there exists $l_1 \in U_1 \cap \tilde{\mathcal{L}}$ such that $\alpha_{l_1} = \beta_1$. In an analogous way one can justify the existence of $l_2 \in U_2 \cap \tilde{\mathcal{L}}$ such that $\alpha_{l_2} = \beta_2$. The above considerations as well as (17) allow us to ascertain the disconnectedness of the set $\tilde{\mathcal{L}}$, which leads to a contradiction with the assumption we have adopted. Condition (15) has thus been shown.

By (15), in view of the definition of the relation $\mathcal{R}_1$, we may infer that N_1 is a set non-disjoint from any abstract class belonging to N^*. Consequently, $g^{c_{x_0}}_{\xi_0}(\tilde{\mathcal{L}}) = \mu(N^*) = \mathcal{S}$. The definition of the function f^* and the inclusion $\tilde{\mathcal{L}} \subset \mathcal{L}_1$ allow one to deduce the veracity of the inclusion $\mathcal{S} \subset f^*(\mathcal{L}_1)$. The connectedness of the set $\mathcal{S}$ and the condition (1) imply the connectedness of the set $f^*(\mathcal{L}_1)$.

II. $\mathcal{L}_1 \cap \mathcal{A} = \emptyset$. Then the following cases may take place:

i) If $\mathcal{L}_1 \subset \mathcal{K}_\xi^{\mathcal{X}}$ for some $\xi \in \Theta$, then $f^*(\mathcal{L}_1)$ is a one-point set.

ii) If $\mathcal{L}_1 \subset \mathcal{K}$, the set $f^*(\mathcal{L}_1) = f(\mathcal{L}_1)$ is connected.

iii) So, let $\mathcal{L}_1 \backslash \mathcal{K} \neq \emptyset$ and $\mathcal{L}_1 \backslash \mathcal{K}_\xi^{\mathcal{X}} \neq \emptyset$ for any $\xi \in \Theta$. There exist $\xi_1, \xi_2 \in \Theta$ satisfying the conditions

$$\xi_1 \neq \xi_2 \qquad \text{and} \qquad \mathcal{L}_1 \cap \mathcal{K}_{\xi_1}^{\mathcal{X}} \neq \emptyset \neq \mathcal{L}_1 \cap \mathcal{K}_{\xi_2}^{\mathcal{X}}. \tag{18}$$

It is not hard to verify that then

$$\mathcal{L}_1 \cap \mathcal{K}_\xi^{\mathcal{X}} \neq \emptyset \qquad \text{for any} \qquad \xi \in (\xi_1, \xi_2). \tag{19}$$

Conditions (18) and (19) as well as the assumptions of case II imply the following relationship:

$$\mathcal{L}_1 \cap (\mathcal{K}_\xi^{\mathcal{X}} \backslash \mathcal{A}) \neq \emptyset \qquad \text{for any} \qquad \xi \in [\xi_1, \xi_2]. \tag{20}$$

In view of (20), we may deduce that the segment $[\xi_1, \xi_2]$ intersects each abstract class belonging to $\hat{N}$, which, in virtue of the definition of the mapping h, means the equality $\mathcal{S} = h(\mathcal{L}_1 \cap (\mathcal{X} \backslash \mathcal{K}))$. The definition of the mapping f^* allows one, in turn, to ascertain the truth of the inclusion $\mathcal{S} \subset f^*(\mathcal{L}_1)$. Consequently, the connectedness of $\mathcal{S}$ and (1) imply the connectedness of the set $f^*(\mathcal{L}_1)$. Thereby, the Darboux property has finally been proved.

At present, we shall show that f^* is a function with finite variation.

For the purpose, consider the Banach indicatrix $\mathcal{N}_{f^*}$ of the function f^*. Let $y \notin \mathcal{S}$. Since $f^*_{|\mathcal{X} \backslash \mathcal{K}} = g \bigtriangledown h$ and $(g \bigtriangledown h)(\mathcal{X} \backslash \mathcal{K}) = \mathcal{S}$, therefore $(f^*)^{-1}(y) \subset \mathcal{K}$. Hence it appears that $(f^*)^{-1}(y) = f^{-1}(y)$. Consequently, $\mathcal{N}_{f^*}(y) = \mathcal{N}_f(y)$ for any $y \notin \mathcal{S}$. The functions $\mathcal{N}_{f^*}$ and $\mathcal{N}_f$ differ from each other on the set $\mathcal{S}$ of measure zero, thus the measurability of $\mathcal{N}_f$ implies the measurability of $\mathcal{N}_{f^*}$, and

$$V(f^*) = \int_{\mathbb{R}^2} \mathcal{N}_{f^*}(y)dy = \int_{\mathbb{R}^2 \backslash \mathcal{S}} \mathcal{N}_f(y)dy \leq \int_{\mathbb{R}^2} \mathcal{N}_f(y)dy < +\infty.$$

Consequently, we have proved that f^* has a finite variation.

We shall demonstrate that

$$Q_f = Q_{f^*}. \tag{21}$$

Let us now observe that

$$f^*(V) \supset \mathcal{S} \text{ for any set } V \text{ open in } \mathcal{X}, \text{ satisfying the condition } V \backslash \mathcal{K} \neq \emptyset. \tag{22}$$

Indeed, let V be the set satisfying the assumptions of condition (22) and let $r \in V \setminus \mathcal{K}$. Since $\mathcal{X}$ is locally stratiformly connected, there exists a neighbourhood V_1 (in $\mathcal{X}$) of the point r, such that $V_1 \subset V \setminus \mathcal{K}$ and $V_{1_{\mathcal{K}}}$ is a connected set. The following two cases are possible:

1^0 Let $V_{1_{\mathcal{K}}}$ be a one-element set and let $V_{1_{\mathcal{K}}} = \{\xi_r\}$. This means that $V_1 \subset \mathcal{K}^{\mathcal{X}}_{\xi_r}$. From the definition of the set $\mathcal{A}$ it follows that $V_1 \subset \mathcal{K}^{\mathcal{X}}_{\xi_r} \cap \mathcal{A}$. Let C be an element of $\mathcal{M}_{\xi_r}$ such that $r \in C$ and let $\delta_r > 0$ be a number for which

$$\overline{K(r,\delta_r)} \cap \mathcal{X} \subset V_1. \tag{23}$$

Let now, for any $z \in C \cap K(r,\delta_r)$, $\alpha_z = m\angle(H_p^{x_C}, H_p^z)$ and let $N_2 = \{\alpha_z : z \in C \cap K(r,\delta_r)\}$. Then

$$\operatorname{Int}_{\mathbb{R}} N_2 \neq \emptyset. \tag{24}$$

Indeed, there are two possible cases.

1. Let $x_C = r$. Then the proof of condition (24) is analogous to that of relation (5), except that in this case the role of ε will be played by the number δ_r.

2. Let $x_C \neq r$. So, let $\mathcal{L}_r$ and $\mathcal{L}'_r$ be the two arcs described in Lemma 2 (for the point r). Then the sets $\mathcal{L}_r \setminus \{r\}$, $\mathcal{L}'_r \setminus \{r\}$ are contained in distinct open half-planes determined by the line the points p and r belong to. Let $a_r \in \mathcal{L}_r$ and $b_r \in \mathcal{L}'_r$ be elements satisfying the conditions: $a_r, b_r \in K(r,\delta_r)$, the angle $\Gamma^* = \angle(H_p^{a_r}, H_p^{b_r})$ contains r and does not contain x_C, and $\mathcal{K}_{\xi_r} \cap \Gamma^* \subset K(r,\delta_r)$.

Note that then

$$\begin{aligned} &H_p^x \cap C \cap K(r,\delta_r) \neq \emptyset \text{ for any half-line } H_p^x \subset \angle(H_p^r, H_p^{a_r}) \\ \text{or } &H_p^x \cap C \cap K(r,\delta_r) \neq \emptyset \text{ for any half-line } H_p^x \subset \angle(H_p^r, H_p^{b_r}). \end{aligned} \tag{25}$$

For let us suppose it is not so. Then there exist two half-lines Λ_1, Λ_2 with the initial point p, contained in $\angle(H_p^r, H_p^{a_r})$ and $\angle(H_p^r, H_p^{b_r})$, respectively, and such that

$$\Lambda_i \cap C \cap K(r,\delta_r) = \emptyset \qquad \text{for} \qquad i = 1, 2, .$$

It is not difficult to deduce from this that $\Lambda_i \cap C = \emptyset$ for $\imath = 1, 2$. Note that the figure $\Lambda_1 \cup \Lambda_2$ cuts the plane $\mathbb{R}^2$ into two domains (r belongs to one of them, while x_C to the other) and is disjoint from C. This fact leads to a contradiction with the connectedness of C. Condition (25) has been proved. This relationship, in turn, implies condition (24).

On the ground of conditions (23), (24) and the method of constructing the functions g_{ξ_r} and f^*, we may write down

$$f^*(V) \supset g_{\xi_r}(V_1) \supset g_{\xi_r}(\overline{K(x,\delta_r)} \cap \mathcal{X}) \supset g_{\xi_r}(C \cap K(r,\delta_r)) = \mu(N^*) = \mathcal{S}.$$

This proves the veracity of condition (22) in this case.

2^0 Let now $V_{1_\mathcal{K}}$ be a non-degenerate interval. If $V_1 \cap \mathcal{A} \neq \emptyset$, then there would exist $\xi \in \Theta'$ such that $W_1 = V_1 \cap \mathcal{K}_\xi^\mathcal{X} \cap \mathcal{A}$ would be, by (4), a non-empty set open in $\mathcal{X}$, and $W_{1_\mathcal{K}} = \{\xi\}$. For W_1, one can carry out an analogous argument as for V_1 in 1^0. So, let $V_1 \cap \mathcal{A} = \emptyset$. Then, in view of the definition of the function h, it follows that

$$f^*(V) \supset h(V_1) = \mathcal{S}.$$

Thereby, the proof of condition (22) has been finished.

The inclusion

$$Q_{f^*} \subset Q_f \tag{26}$$

is a consequence of this condition.

Indeed, note first that $Q_{f^*} \subset \mathcal{K}$ and let $x_0 \in Q_{f^*}$. We want to show the relation $x_0 \in Q_f$. Let U be an open neighbourhood of the point x_0 in $\mathcal{K}$ and let $V = K(f(x_0),\delta)$ where $\delta > 0$. Then there exists a neighbourhood U' of the point x_0 in $\mathcal{X}$ such that $U = U' \cap \mathcal{K}$. Since $x_0 \in Q_{f^*}$, and V is an open neighbourhood of the point $f^*(x_0) = f(x_0)$ in $\mathbb{R}^2$, there exists a non-empty set $W \subset U'$ open in $\mathcal{X}$, such that

$$f^*(W) \subset V. \tag{27}$$

If $W \backslash \mathcal{K} \neq \emptyset$, then, by (22), $f^*(W) \supset \mathcal{S}$ and condition (27) would not be satisfied. The set $W \subset \mathcal{K} \cap U' = U$ is a non-empty set open in $\mathcal{K}$ and, by virtue of (27), $x_0 \in Q_f$. We shall further demonstrate that

$$Q_f \subset Q_{f^*}. \tag{28}$$

Indeed, let $x_0 \in Q_f$, let U be an open neighbourhood of the point x_0 in $\mathcal{X}$ and let V be an open neighbourhood of the point $f^*(x_0) = f(x_0)$ in $\mathbb{R}^2$. Then $U' = U \cap \mathcal{K}$ is an open (in $\mathcal{K}$) neighbourhood of the point x_0. Consequently, there exists a set W' open in $\mathcal{K}$, such that $W' \subset U'$ and $f(W') \subset V$. To prove (28), it is sufficient to show that $\mathrm{Int}_\mathcal{X} W' \neq \emptyset$. For the purpose, it suffices to notice, in turn, that $\mathrm{Int}_{\mathbb{R}^2} W' \neq \emptyset$.

On account of the fact that $p \in \mathrm{Int}_{\mathbb{R}^2} \mathcal{K}$, we may deduce the existence of $q \in W'$ such that $q \neq p$. Let $\gamma > 0$ be a real number such that $K(q,\gamma) \cap \mathcal{K} \subset$

W'. Of course, $p \in \mathrm{Int}_{\mathbb{R}^2} \mathcal{K}$, therefore to a line M perpendicular to H_p^q, such that $p \in M$, there belongs some element $p_1 \in \mathcal{K}\setminus\{p\}$. By virtue of the convexity of $\mathcal{K}$, the triangle $\Delta(p, p_1, q) \subset \mathcal{K}$. Consequently, the non-empty set $\Delta(p, p_1, q) \cap K(q, \gamma)$ is contained in the set W' and possesses a non-empty interior in $\mathbb{R}^2$. Condition (28) has been shown.

Relationships (26) and (28) imply equality (21).

Now, we shall show that

$$Q_f \subset \tilde{Q}_{f^*}(\mathcal{K}). \tag{29}$$

Indeed, let $x \in Q_{f^*}$. Relationship (21) implies the relation $x \in \mathcal{K}$. Let $V = K(f^*(x), 1)$ and let W be an neighbourhood of the point x open in $\mathcal{X}$. It should be demonstrated that $\mathrm{Int}_{\mathcal{X}}(W \cap (f^*)^{-1}(V)) \subset \mathcal{K}$. For if it were not so, then the set $\mathrm{Int}_{\mathcal{X}}(W \cap (f^*)^{-1}(V))\setminus\mathcal{K} \neq \emptyset$ would be open (in $\mathcal{X}$) and, by (22), the inclusion

$$f^*((\mathrm{Int}_{\mathcal{X}}(W \cap (f^*)^{-1}(V)))\setminus\mathcal{K}) \supset \mathcal{S}$$

would take place.

However, on the other hand,

$$f^*(\mathrm{Int}_{\mathcal{X}}(W \cap (f^*)^{-1}(V))) \subset f^*((f^*)^{-1}(V)) \subset V.$$

This contradicts the definitions of $\mathcal{S}$ and V. Condition (29) has been shown and, thereby, the proof of the theorem has been completed.

A natural problem appearing after the analysis of Theorem 1 is the question concerning the possibility of extending a Darboux function with finite variation in such a way that the terminal transformation should possess as many points of continuity and quasi-continuity as possible. The answer to this question is included in the following theorem.

Theorem 2 *Let $\mathcal{K}$ be a closed non-empty subset of the plane $\mathbb{R}^2$ and let $f : \mathcal{K} \to \mathbb{R}^2$ be a Darboux function with finite variation. Then there exists a Darboux function $f^* : \mathbb{R}^2 \to \mathbb{R}^2$ with finite variation such that $f^*_{|\mathcal{K}} = f$ and*

$$Q_{f^*} = Q_f \cup \overline{\mathbb{R}^2\setminus\mathcal{K}} \qquad \textit{and} \qquad \mathcal{C}_{f^*} \supset \mathbb{R}^2\setminus\mathcal{K}.$$

PROOF. If $\mathcal{K} = \mathbb{R}^2$, then the function $f^* = f$ satisfies the conditions of the assertion of the theorem. So, let $\mathcal{K} \neq \mathbb{R}^2$. Then there exists $x' \in \mathbb{R}^2\setminus\mathcal{K}$. Denote $d' = d(\mathcal{K}, x')$. In view of the separability of the space $\mathbb{R}^2$, it follows that there exists a set $\{q_n : n \in \mathbb{N}\}$ dense in $\mathbb{R}^2$. Let $L_n = [q_n, q_{n+1}]$ for any $n \in \mathbb{N}$ and let the mapping

$$h_n : \left[\frac{d'}{n+1}, \frac{d'}{n}\right] \xrightarrow{\text{onto}} L_n \subset \bigcup_{i \in \mathbb{N}} L_i \subset \mathbb{R}^2$$

be a homeomorphism satisfying the condition

$$h_n\left(\frac{d'}{j}\right) = q_j \tag{1}$$

for $j = n, n+1$ and $n = 1, 2, \ldots$.
Moreover, let

$$h_0(x) = q_1 \qquad \text{for} \qquad x \in [d', +\infty).$$

It is easy to show that combination

$$h = \underset{n\in\mathbb{N}\cup\{0\}}{\nabla h_n} : (0, +\infty) \to \mathbb{R}^2$$

is a continuous function.

Consider a mapping $f^* : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula

$$f^*(x) = \begin{cases} h(d(\mathcal{K}, x)) & \text{when } x \notin \mathcal{K} \\ f(x) & \text{when } x \in \mathcal{K}. \end{cases}$$

We shall demonstrate that f^* has the Darboux property.

Let $\mathcal{L}$ be an arc contained in $\mathbb{R}^2$. If $\mathcal{L} \subset \mathcal{K}$, then the set $f^*(\mathcal{L}) = f(\mathcal{L})$ is, of course, connected. So, let $\mathcal{L}\backslash\mathcal{K} \neq \emptyset$. Then the following two cases are possible:

1[0] Let $\mathcal{L} \cap \mathcal{K} = \emptyset$. Then $f^*(\mathcal{L}) = (h \circ d_{\mathcal{K}})(\mathcal{L})$ where $d_{\mathcal{K}}$ stands for the function $\mathbb{R}^2 \ni x \mapsto d(\mathcal{K}, x) \in [0, +\infty)$. Since h and $d_{\mathcal{K}}$ are continuous transformations and $\mathcal{L}$ is a connected set, therefore $f^*(\mathcal{L})$ is a connected set, too.

2[0] Let $\mathcal{L} \cap \mathcal{K} \neq \emptyset$. Denote by n_0 a positive integer such that $\frac{d'}{n_0} \in d_{\mathcal{K}}(\mathcal{L})$. Evidently, $0 \in d_{\mathcal{K}}(\mathcal{L})$. By virtue of of the continuity of $d_{\mathcal{K}}$ and the connectedness of $\mathcal{L}$ as well as the condition $d_{\mathcal{K}}(\mathcal{K}) = \{0\}$, the inclusion

$$\left(0, \frac{d'}{n_0}\right] \subset d_{\mathcal{K}}(\mathcal{L}\backslash\mathcal{K})$$

follows.

The above condition allows one to prove the following relationship:

$$f^*(\mathcal{L}) \supset \bigcup_{n \geq n_0} L_n. \tag{2}$$

Indeed,

$$f^*(\mathcal{L}) \supset f^*(\mathcal{L}\backslash\mathcal{K}) = f^*_{|\mathbb{R}^2\backslash\mathcal{K}}(\mathcal{L}\backslash\mathcal{K}) = h(d_{\mathcal{K}}(\mathcal{L}\backslash\mathcal{K}))$$
$$\supset h\left(\left(0, \frac{d'}{n_0}\right]\right) = \bigcup_{n\geq n_0} L_n.$$

Since $\bigcup_{n\geq n_0} L_n \supset \{q_n : n \geq n_0\}$ and the set $\{q_n : n \geq n_0\}$ is dense in $\mathbb{R}^2$, therefore

$$\overline{\bigcup_{n\geq n_0} L_n} = \mathbb{R}^2. \tag{3}$$

Of course, $\bigcup_{n\geq n_0} L_n$ is a connected set, which, in view of (2) and (3), allows one to infer that $f^*(\mathcal{L})$ is a connected set, too.

Now, we shall show that f^* is a function with finite variation. So, let $\mathcal{N}_f$ and $\mathcal{N}_{f^*}$ denote the Banach indicatrices of the functions f and f^*, respectively. It can easily be noticed that $m_2(\bigcup_{n\in\mathbb{N}} L_n) = 0$. This means that the functions $\mathcal{N}_f$ and $\mathcal{N}_{f^*}$ differ from each other on at most the set of measure zero. Consequently, the function $\mathcal{N}_{f^*}$ is measurable and f^* has a finite variation since the functions $\mathcal{N}_f$ and f, respectively, have such properties.

Note now that

$$\mathbb{R}^2\backslash\mathcal{K} \subset C_{f^*}. \tag{4}$$

Indeed, let $x \in \mathbb{R}^2\backslash\mathcal{K}$ and let $\varepsilon > 0$. Then there exists $\delta_1 > 0$ such that $K(x, \delta_1) \cap \mathcal{K} = \emptyset$. Since $h \circ d_{\mathcal{K}}$ is a continuous function, therefore

$$(h \circ d_{\mathcal{K}})(K(x, \delta)) \subset K((h \circ d_{\mathcal{K}})(x), \varepsilon)$$

for some $\delta \in (0, \delta_1)$. In view of the equality $f^*_{|\mathbb{R}^2\backslash\mathcal{K}} = h \circ d_{\mathcal{K}}$, condition (4) is proved.

We shall show that

$$Q_{f^*} = Q_f \cup \overline{\mathbb{R}^2\backslash\mathcal{K}}. \tag{5}$$

For the purpose, we shall first prove the inclusion

$$Q_{f^*} \supset Q_f \cup \overline{\mathbb{R}^2\backslash\mathcal{K}}. \tag{6}$$

Let $x \in Q_f \cup \overline{\mathbb{R}^2\backslash\mathcal{K}} = (Q_f \cap \operatorname{Int}\mathcal{K}) \cup \overline{\mathbb{R}^2\backslash\mathcal{K}}$. It should be demonstrated that $x \in Q_{f^*}$. Let $U \ni f^*(x)$ and $V \ni x$ be open sets in $\mathbb{R}^2$. The following three cases are possible:

i) $x \in Q_f \cap \operatorname{Int} \mathcal{K}$. Then $V_1 = V \cap \operatorname{Int} \mathcal{K}$ is an open set in $\mathcal{K}$ and in $\mathbb{R}^2$. In view of the quasi-continuity of the function f at the point x,

$$f(W) = f^*(W) \subset U \tag{7}$$

for some subset W of the set V_1, open in $\mathcal{K}$. The openness of the set W in $\mathcal{K}$ implies the existence of a set W_1 open in $\mathbb{R}^2$ such that $W = W_1 \cap \mathcal{K}$. The last condition and the inclusion $W \subset V_1$ allow one to deduce that $W = W_1 \cap V_1$, which, in view of the openness of the sets V_1 and W_1 in $\mathbb{R}^2$, means that the set W is open in $\mathbb{R}^2$. Consequently, the relation $x \in Q_{f^*}$ follows from (7) and the inclusion $W \subset V_1 \subset V$.

ii) $x \in \mathbb{R}^2 \backslash \mathcal{K}$. Then $x \in Q_{f^*}$ by (4).

iii) $x \in \operatorname{Fr} \mathcal{K} = \operatorname{Fr}(\mathbb{R}^2 \backslash \mathcal{K})$. Let Z stand for some open ball with centre at a point x, contained in V. Of course, $Z \backslash \mathcal{K} \neq \emptyset$. In view of the connectedness of $\mathcal{K}$, $x \in \mathcal{K} \cap Z$. Consequently, in virtue of the assumptions concerning the number d' and the connectedness of Z, there exists a number $n_1 \in \mathbb{N}$ such that for any number $r \leq \frac{d'}{n_1}$, there exists an element $x_r \in Z$ satisfying the condition $d_{\mathcal{K}}(x_r) = r$. Let $n_2 \geq n_1$ be a positive integer so chosen that $q_{n_2} \in U$ (the existence of such a number follows from the density of the set $\{q_n : n \geq n_1\}$ in $\mathbb{R}^2$). It is easy to check that the element $x_0 = x_{\frac{d'}{n_2}}$ has the following properties:

a) $x_0 \in Z \backslash \mathcal{K}$,

b) $x_0 \in \mathcal{C}_{f^*}$,

c) $f^*(x_0) = q_{n_2} \in U$.

In view of these facts, there exists a set W open in $\mathbb{R}^2$, such that $W \subset Z \backslash \mathcal{K} \subset V$ and $f^*(W) \subset U$. The proof of condition (6) has thus been completed.

We shall now show that

$$Q_{f^*} \subset Q_f \cup \overline{\mathbb{R}^2 \backslash \mathcal{K}}. \tag{8}$$

Let $x \in Q_{f^*}$ and suppose that $x \notin Q_f \cup \overline{\mathbb{R}^2 \backslash \mathcal{K}}$. Then $x \in \operatorname{Int} \mathcal{K}$. Let $\zeta > 0$ and $\eta > 0$. Then there exists $\eta_1 < \eta$ such that $K(x, \eta_1) \subset \mathcal{K}$. In view of the quasi-continuity of the function f^* at the point x, we may infer that there exists a ball $K(y, \eta_2) \subset K(x, \eta_1)$ such that $f^*(K(y, \eta_2)) \subset K(f^*(x), \zeta)$. Of course, $K(y, \eta_2) \subset K(x, \eta) \cap \mathcal{K}$, therefore $f(K(y, \eta_2)) \subset K(f(x), \zeta)$, too, which proves that $x \in Q_f$. This fact contradicts our assumption, and thus, condition (8) has been proved.

Conditions (6) and (8) imply (5).

Thereby, the proof of the theorem has finally been completed.

References

[1] R. J. Pawlak, *Extensions of Darboux functions*, Real Analysis Exchange, **15** (1989–90), p. 511–547.