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## POINTS OF NON-DIFFERENTIABILITY OF TYPICAL LIPSCHITZ FUNCTIONS

### Abstract

We show that typical Lipschitz functions on  $[0, 1]$  (more precisely, typical functions in the space of functions on  $[0, 1]$  with Lipschitz constant at most one endowed with the topology of uniform convergence) have much better differentiability behaviour than general Lipschitz functions.

While the Lebesgue Theorem, according to which real-valued Lipschitz functions defined on the real line are differentiable almost everywhere, gives a complete answer to the question of size of sets of points of non-differentiability of such functions, its higher dimensional extension, the Rademacher Theorem, does not describe the full story. More precisely, for every  $E \subset \mathbb{R}$  of Lebesgue measure zero one can find a Lipschitz function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is non-differentiable at every point of  $E$  (see [1] or, for a complete characterization of sets of non-differentiability of Lipschitz functions, [7]), but there is  $E \subset \mathbb{R}^2$  of Lebesgue measure zero such that every Lipschitz function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  has a point of differentiability inside  $E$  (see [5]). The higher dimensional situation is full of puzzling open problems out of which our favourite is the question what happens if we consider pairs of Lipschitz functions instead of one function. In other words, is it true that for every set  $E \subset \mathbb{R}^2$  of Lebesgue measure zero one can find a Lipschitz mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is non-differentiable at every point of  $E$ ?

Using the “argument” that typical functions have the worst differentiability behaviour, one may hope that this problem may be solved using the Baire Category Theorem in a suitable space of Lipschitz functions on, say, a square. Two such spaces seem to be reasonable candidates: The space of all Lipschitz

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functions with the norm given by the Lipschitz constant and the space of Lipschitz functions with uniformly bounded Lipschitz constant (say, by one) in the topology of uniform convergence. However, for our purposes, the former space appears to have too many open sets; for example, it is non-separable, has no simple dense subsets, and differentiable functions form a nowhere dense closed set (and therefore cannot be used as approximating functions). Though the latter space has much better properties from this point of view, we show here that it has another disadvantage: It cannot be used in the way suggested because typical functions from it have much better differentiability behaviour than general Lipschitz functions. We will show this for Lipschitz functions on the interval  $[0, 1]$  (instead of on the square); though this simplification is not necessary, it avoids some technical complications and makes even the formulation of the problem much clearer: Given a set  $E \subset [0, 1]$  of Lebesgue measure zero, is it true that typical Lipschitz functions  $f : [0, 1] \mapsto \mathbb{R}$  are non-differentiable at every point of  $E$ ? Our answer follows immediately from Lemma 2 (or, with little work, from the Theorem): Every residual subset of  $[0, 1]$  of measure zero contains points of differentiability of typical functions from this space.

We shall therefore consider the space  $\text{Lip}_1$  of all functions  $f : [0, 1] \mapsto \mathbb{R}$  verifying  $|f(y) - f(x)| \leq 1$  for all  $x, y \in [0, 1]$  (i.e., with Lipschitz constant not exceeding one) equipped with the topology of uniform convergence (which, in this case, is the same as the topology of pointwise convergence). Then  $\text{Lip}_1$  is a complete (separable) metric space in which we answer the question of differentiability properties of typical functions in the following way.

**Theorem** *Let  $E$  be an analytic subset of  $[0, 1]$ . Then the following statements are equivalent.*

- (i) *The set of those functions  $f \in \text{Lip}_1$  which are differentiable at no point of  $E$  is residual in  $\text{Lip}_1$ .*
- (ii) *The set  $E$  is contained in an  $F_\sigma$  subset of  $[0, 1]$  of measure zero.*

To prove the Theorem we first introduce some notation, remind ourselves of the Banach-Mazur game, and prove two preparatory lemmas.

We shall denote by  $\mathcal{P}$  the family of all functions  $f \in \text{Lip}_1$  which are piecewise affine and satisfy  $|f'(x)| = 1$  at every point at which the derivative exists. It is easy to see that  $\mathcal{P}$  is a dense subset of  $\text{Lip}_1$ . By the word measure we shall mean Lebesgue measure on  $[0, 1]$  and we shall denote it by the sign of absolute value. A measurable set  $F \subset [0, 1]$  is said to have every portion of positive measure if for each open interval  $I \subset \mathbb{R}$  the intersection  $I \cap F$  is of positive measure provided it is non-empty.

In some situations it is more convenient to deduce the residuality of a set by using the *Banach-Mazur game* which is, in full generality, defined as follows.

Let  $X$  be a topological space,  $S$  a subset of  $X$  and let two players, Player I and Player II, play the game in which a play is a non-increasing sequence of non-empty open sets

$$U_1 \supset V_1 \supset U_2 \supset V_2 \supset \dots$$

which have been chosen alternately; the  $U_k$ 's by Player I, the  $V_k$ 's by Player II. Player II is said to have won a play provided  $\bigcap_{k=1}^{\infty} V_k \subset S$ ; otherwise, Player I has won. We say that Player II has a winning strategy if, using it, she wins every play independently of Player I's choices.

The connection with residuality of  $S$  (which, in this generality, means that there are dense open subsets  $G_k$  of  $X$  such that  $S \supset \bigcap_{k=1}^{\infty} G_k$ ) is given in the following (the proof from [3] may be easily generalized, or see [6, Theorem 4.23]):

**Proposition.** *Player II has a winning strategy if and only if  $S$  is a residual subset of  $X$ .*

**Lemma 1.** *Suppose that  $f \in \mathcal{P}$  and that  $0 < \varepsilon \leq 1$ . Then there exists an open neighbourhood  $V$  of  $f$  in  $\text{Lip}_1$  with  $\text{diam}(V) < \varepsilon$  such that for every  $g \in V \cap \mathcal{P}$  one can find an open set  $G \subset (0, 1)$  having the following properties:*

- (1)  $|[0, 1] \setminus G| < \varepsilon$ ,
- (2)  $(g - f)'(x) = 0$  at every  $x \in G$ , and
- (3)  $|(g(y) - g(x)) - (f(y) - f(x))| \leq \varepsilon|y - x|$  whenever  $x \in G$  and  $y \in [0, 1]$ .

**PROOF.** We show that the statement holds with

$$V = \left\{ g \in \text{Lip}_1 : \max\{|g(x) - f(x)| : x \in [0, 1]\} < \frac{\varepsilon^3}{64(N+1)} \right\},$$

where  $N$  is the number of points at which  $f$  is not differentiable;  $V$  is clearly an open neighbourhood of  $f$  verifying  $\text{diam}(V) < \varepsilon$ .

Whenever  $g \in V$  and  $I \subset [0, 1]$  is an interval on which  $f$  is affine, we use that either  $f'(x) = 1$  for all  $x \in I$  or  $f'(x) = -1$  for all  $x \in I$  to deduce that the function  $g' - f'$  does not change its sign on  $I$ . Hence

$$\begin{aligned} \int_I |g'(x) - f'(x)| dx &= \left| \int_I (g'(x) - f'(x)) dx \right| \\ &\leq 2 \max\{|g(x) - f(x)| : x \in [0, 1]\} \\ &< \frac{\varepsilon^3}{32(N+1)}. \end{aligned}$$

Since almost all of  $[0, 1]$  is a union of  $N + 1$  such intervals, we conclude that

$$\int_0^1 |g'(x) - f'(x)| \, dx < \frac{\varepsilon^3}{32}.$$

Let  $g \in V \cap \mathcal{P}$ . Let  $F_0$  be the (finite) set of all points at which at least one of the functions  $f, g$  is not differentiable, let  $F_1$  be the set of those points  $x \in [0, 1] \setminus F_0$  at which  $f'(x) \neq g'(x)$ , and let  $F_2$  be the set of those points  $x \in [0, 1] \setminus (F_0 \cup F_1)$  for which one can find  $y \in [0, 1]$  such that  $y \neq x$  and  $|(g(y) - g(x)) - (f(y) - f(x))| \geq \varepsilon|y - x|$ .

Since  $f, g \in \mathcal{P}$ , the sets  $F_0, F_0 \cup F_1, F_0 \cup F_1 \cup F_2$  are closed,  $|F_0| = 0$ , and  $|g'(x) - f'(x)| = 2$  for every  $x \in F_1$ . Hence

$$|F_1| \leq \frac{1}{2} \int_0^1 |g'(x) - f'(x)| \, dx < \varepsilon/2.$$

To estimate the measure of the set  $F_2$ , we recall from, e.g., [2, 21.76(i), p. 424] that the Hardy-Littlewood maximal operator  $\mathbf{M}\varphi$  which is defined for measurable functions  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  by

$$\mathbf{M}\varphi(x) = \sup \left\{ \frac{1}{|I|} \int_I |\varphi(x)| \, dx : I \text{ is an interval containing } x \right\}$$

verifies the inequality

$$\int (\mathbf{M}\varphi(x))^2 \, dx \leq 8 \int \varphi^2(x) \, dx.$$

Since  $\mathbf{M}(g' - f')(x) \geq \varepsilon$  for every  $x \in F_2$ , we obtain

$$\begin{aligned} |F_2| &\leq \varepsilon^{-2} \int (\mathbf{M}(g' - f')(x))^2 \, dx \leq 8\varepsilon^{-2} \int (g' - f')^2(x) \, dx \\ &\leq 16\varepsilon^{-2} \int |g' - f'| \, dx < \varepsilon/2. \end{aligned}$$

Consequently,  $|F_0 \cup F_1 \cup F_2| < \varepsilon$ , and all the statements of the lemma hold with  $G = (0, 1) \setminus (F_0 \cup F_1 \cup F_2)$ .

**Lemma 2.** *Let  $F \subset [0, 1]$  be a non-empty closed set with every portion of positive measure and let  $E$  be such that  $E \cap F$  is residual in  $F$ . Then the set  $S$  of those functions from  $\text{Lip}_1$  which are differentiable at least at one point of  $E \cap F$  is residual in  $\text{Lip}_1$ .*

PROOF. Let  $E_1 \supset E_2 \supset \dots$  be relatively open dense subsets of  $F$  such that  $\bigcap_{k=1}^{\infty} E_k \subset E$ . To prove that the set  $S$  is residual in  $\text{Lip}_1$ , we describe a winning strategy in the corresponding Banach-Mazur game in which, in addition to the non-empty open subsets  $V_k$  of  $\text{Lip}_1$ , Player II is choosing functions  $f_k \in V_k \cap \mathcal{P}$  and non-empty relatively open subsets  $M_k$  of  $F$  in such a way that the following properties are verified:

- (1)  $\text{diam}(V_k) \leq 2^{-k+1}$ .
- (2) For every  $g \in V_k \cap \mathcal{P}$  there is an open set  $G \subset (0, 1)$  such that
  - (a)  $|[0, 1] \setminus G| < |M_k \cap E_k|$ ,
  - (b)  $(g - f_k)'(x) = 0$  at every  $x \in G$ , and
  - (c)  $|(g(y) - g(x)) - (f_k(y) - f_k(x))| \leq 2^{-k}|y - x|$  if  $x \in G, y \in [0, 1]$ .
- (3) At every point  $x$  of  $M_k$  the functions  $f_1, f_2, \dots, f_k$  are differentiable and  $f'_1(x) = f'_2(x) = \dots = f'_k(x)$ .
- (4)  $\overline{M_k} \subset M_{k-1} \cap E_{k-1}$  if  $k \geq 2$ .
- (5)  $|(f_k(y) - f_k(x)) - (f_{k-1}(y) - f_{k-1}(x))| \leq 2^{-k+1}|y - x|$  whenever  $k \geq 2, x \in M_k$ , and  $y \in [0, 1]$ .

The required strategy for Player II can be described as follows: The construction of the answer of Player II to the first move  $U_1$  of Player I starts by picking an arbitrary  $f_1 \in U_1 \cap \mathcal{P}$ . Then  $M_1$  is defined as the set of points of  $F$  at which  $f_1$  is differentiable; this clearly verifies (3). Since  $F \setminus M_1$  is finite,  $M_1$  is non-empty and relatively open in  $F$ . Since  $E_1$  is a dense relatively open subset of  $F$ , the set  $M_1 \cap E_1$  is a non-empty relatively open subset of  $F$  and therefore has positive measure. Hence Lemma 1 with  $f = f_1$  and  $\varepsilon = \min\{1, |M_1 \cap E_1|\}$  gives an open neighbourhood  $V_1$  of  $f_1$  such that (1) and (2) hold. This choice verifies all our requirements, since the remaining ones, (4) and (5), do not concern the case  $k = 1$ .

If  $k \geq 2$  and open sets  $\text{Lip}_1 \supset U_1 \supset V_1 \supset \dots \supset U_{k-1} \supset V_{k-1}$ , functions  $f_1, \dots, f_{k-1}$ , and non-empty relatively open subsets  $M_1, \dots, M_{k-1}$  of  $F$  verifying the above conditions have been already defined and if  $U_k \subset V_{k-1}$  has been the next move of Player I, then Player II chooses an arbitrary  $f_k \in U_k \cap \mathcal{P}$  and uses (2) with  $k$  replaced by  $k - 1$  and with  $g = f_k$  to find an open set  $G \subset (0, 1)$  such that

- ( $\alpha$ )  $|[0, 1] \setminus G| < |M_{k-1} \cap E_{k-1}|$ ,
- ( $\beta$ )  $(f_k - f_{k-1})'(x) = 0$  at every  $x \in G$ , and

$$(\gamma) |(f_k(y) - f_k(x)) - (f_{k-1}(y) - f_{k-1}(x))| \leq 2^{-k+1}|y - x| \text{ if } x \in G \text{ and } y \in [0, 1].$$

Since the set  $G \cap M_{k-1} \cap E_{k-1}$  is relatively open in  $F$  and since  $(\alpha)$  implies that it is non-empty, there is a non-empty relatively open subset  $M_k$  of  $F$  such that  $\overline{M_k} \subset G \cap M_{k-1} \cap E_{k-1}$ . Because of  $(\alpha-\gamma)$ , this choice verifies (3-5). The remaining part of the construction is similar to the case  $k = 1$ : Since  $E_k$  is a dense relatively open subset of  $F$ , the set  $M_k \cap E_k$  is a non-empty relatively open subset of  $F$  and therefore has positive measure. Hence Lemma 1 with  $f = f_k$  and  $\varepsilon = \min\{2^{-k}, |M_k \cap E_k|\}$  gives an open neighbourhood  $V_k$  of  $f_k$  such that (1) and (2) hold.

It remains to show that any function  $f \in \bigcap_{k=1}^{\infty} V_k$  is differentiable at some point of  $E \cap F$ . Because of (1), there is at most one such function  $f$  and  $f_k \rightarrow f$ . In view of (4),  $\emptyset \neq \bigcap_{k=1}^{\infty} M_k \subset E$ , so it suffices to show that  $f$  is differentiable at every  $x \in \bigcap_{k=1}^{\infty} M_k$ . To prove this, we use, for any  $m = 1, 2, \dots$ , the inequality (5) together with  $f'_m(x) = f'_1(x)$  (which follows from (3)) to estimate

$$\begin{aligned} \left| \frac{f(y) - f(x)}{y - x} - f'_1(x) \right| &\leq \left| \frac{f_m(y) - f_m(x)}{y - x} - f'_m(x) \right| \\ &\quad + \sum_{k=m}^{\infty} \left| \frac{f_k(y) - f_k(x)}{y - x} - \frac{f_{k+1}(y) - f_{k+1}(x)}{y - x} \right| \\ &\leq \left| \frac{f_m(y) - f_m(x)}{y - x} - f'_m(x) \right| + 2^{-m+1}. \end{aligned}$$

Consequently,

$$\limsup_{y \rightarrow x} \left| \frac{f(y) - f(x)}{y - x} - f'_1(x) \right| \leq 2^{-m+1},$$

which shows that  $f'(x) = f'_1(x)$ .

**Proof of the Theorem.** Suppose first that (ii) holds. It clearly suffices to assume that  $E$  is a non-empty closed set of measure zero. Let  $G_k$  be the set of those  $f \in \text{Lip}_1$  for which one can find  $\delta > 0$  with the property that for every  $x \in E$  there is  $y \in [0, 1]$  such that  $\delta < |y - x| < \frac{1}{k}$  and

$$\frac{f(y) - f(x)}{y - x} > 1 - \frac{1}{k} + \delta.$$

Clearly,  $G_k$  are open subsets of  $\text{Lip}_1$ , so to prove that  $G = \bigcap_{k=1}^{\infty} G_k$  is a residual subset of  $\text{Lip}_1$  it suffices to show that it is dense. Whenever  $f \in \text{Lip}_1$ , let

$$f_j(x) = f(0) + \int_0^x \varphi_j(t) dt,$$

where  $\varphi_j(t) = f'(t)$  if  $\text{dist}(t, E) > 1/j$  and  $\varphi_j(t) = 1$  if  $\text{dist}(t, E) \leq 1/j$ . Since  $E$  is a closed subset of  $[0, 1]$ ,

$$\lim_{j \rightarrow \infty} |\{t \in [0, 1] : \text{dist}(t, E) \leq 1/j\}| = 0.$$

Hence  $f_j \rightarrow f$ , which, since clearly  $f_j \in G$ , shows that  $G$  is a residual subset of  $\text{Lip}_1$ . Thus the set  $S = G \cap \{f \in \text{Lip}_1 : -f \in G\}$  is also residual in  $\text{Lip}_1$ . Using the definition of  $G$  to infer that every  $f \in S$  verifies

$$\limsup_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = 1 > -1 = \liminf_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}$$

for every  $x \in E$ , we conclude that the functions from  $S$  are differentiable at no point of  $E$ , which proves (i).

Suppose now that (ii) fails, i.e., that  $E$  cannot be covered by any  $F_\sigma$  set of measure zero. Using [4], we find a closed non-empty subset  $F$  of  $[0, 1]$  with every portion of positive measure such that  $E \cap F$  is residual in  $F$ . Applying Lemma 2, we infer that typical functions have points of differentiability inside  $E$ , which shows that (i) fails as well.

**Remark.** The basic ingredient of the proof of the Theorem is that the situation when  $|f'(x)| = 1$  is extremal in the space of functions with Lipschitz constant not exceeding one. Interestingly enough, this phenomenon is also responsible for the main motivation behind our investigation, namely for the existence of a subset of  $\mathbb{R}^2$  of Lebesgue measure zero inside which every Lipschitz  $f : \mathbb{R}^2 \mapsto \mathbb{R}$  has a point of differentiability.

Another manifestation of the same phenomenon is that typical functions in  $\text{Lip}_1$  verify, at every  $x \in [0, 1]$ ,

$$\limsup_{y \rightarrow x} \left| \frac{f(y) - f(x)}{y - x} \right| = 1.$$

This is easy to see, since the sets  $H_k$  of those  $f \in \text{Lip}_1$  for which one can find  $\delta > 0$  with the property that for every  $x \in [0, 1]$  there is  $y \in [0, 1]$  such that  $\delta < |y - x| < \frac{1}{k}$  and  $|f(y) - f(x)| > (1 - \frac{1}{k} + \delta)|y - x|$  are open and contain  $\mathcal{P}$ . Hence  $\bigcap_{k=1}^{\infty} H_k$  is a residual subset of  $\text{Lip}_1$  consisting only of functions having the above property.

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