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MULTIPLIERS FOR SOME GENERALIZED RIEMANN INTEGRALS IN THE REAL LINE

Let $\mathbb{R}$ be the real line. If $E \subset \mathbb{R}$, then $|E|$ and $d(E)$ respectively denote the Lebesgue measure and the diameter of E. An *interval* is always a compact nondegenerate subinterval of $\mathbb{R}$. A *figure* on an interval $A=[a,b]$ is, by definition, a finite nonempty union of subintervals of A. A collection of figures is called *nonoverlapping* whenever their interiors are disjoint. All functions we consider are real-valued.

Let F be a function on an interval A and let B be a figure on A with n connected components $[a_1,b_1],\dots,[a_n,b_n]$. Then we set $||B||= 2n$ and $F(B) = \sum_{h=1}^{n}[F(b_h) - F(a_h)]$. In particular $F([a,b]) = F(b) - F(a)$. This notation leads to no confusion as the notion of the image of a set under a function is never used this paper.

The *regularity of B with respect to a point $x \in \mathbb{R}$* is the number

$$r(B,x) = \frac{|B|}{d(B \cup \{x\})\ ||B||}\ .$$

If $r(B,x) > \varepsilon > 0$, the figure B is called *ε-regular* with respect to x.

For each subset J of $\{1,2,\dots,n\}$ the set $\cup_{j\in J}[a_j,b_j]$ will be called a *subfigure of B*.

A *partition in A* is a collection (possibly empty) $\{(A_1,x_1),\dots,(A_p,x_p)\}$ where $A_1,\dots,A_p$ are nonoverlapping figures on A and $x_1,\dots,x_p$ are points of A. If $\cup_h A_h = A$, then the partition is called a *partition of A*.

A *gage on A* is a positive function δ defined on A. Let $\varepsilon > 0$ and let δ be a gage on A. A partition $\{(A_h,x_h) : h = 1,\dots,p\}$ in A is called:

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1) *special* if A_h is an interval for $h = 1, \dots, p$;

2) *tight* if $x_h \in A_h$ for $h = 1, \dots, p$;

3) *ε-regular* if $r(A_h, x_h) > \varepsilon$ for $h = 1, \dots, p$;

4) *δ-fine* if $d(A_h \cup x_h) < \delta(x_h)$ for $h = 1, \dots, p$.

In [**BGP**] the following definition was introduced.

Definition 1 *We say that a function f on A is R_o^*-integrable on A if there is a real number I which satisfies the following condition: given $\varepsilon > 0$, there exists a gage δ such that $|\sum_{h=1}^{p} f(x_h)|A_h| - I| < \varepsilon$ for each ε-regular δ-fine partition $\{(A_h, x_h) : h = 1, \dots, p\}$ of A. If the inequality holds only when the partition is also special or tight or special and tight, we say that f is, respectively, R_s^*- or R_t^*- or R_{st}^*-integrable on A.*

It is clear that the R_{st}^*-integral is the usual Henstock integral (H-integral) which in turn coincides with the classical Denjoy-Perron integral (see [**H**] or [**G**]). It is proved in [**BGP**] that the R_o^*-, R_s^*- and R_t^*-integrals are properly included in the H-integral.

It is known (see [**L**] and [**S**]) that for each H-integrable function f and for each function g of bounded variation, the function fg is also H-integrable and the integration by parts formula holds:

$$(1) \qquad (H)\int_a^b fg\,dt = [Fg]_a^b - (L)\int_a^b F\,dg$$

where $F(x) = (H)\int_a^x f\,dt$.

In this paper we prove that any function g of bounded variation is a multiplier also for the families of R_o^*-, R_s^*- or R_t^*-integrable functions. For g a Lipschitz function this problem was considered in [**BGP**, Corollary 4.3 and Remark 4.4] and in the multidimensional case in [**MP**]. The question of whether our present result can be extended to the multidimensional case is open.

We need the following lemmas.

Lemma 1 (See [**BGP**, Lemma 2.12].) *Let Φ be a function on A which has a finite derivative $\Phi'(x)$ at an interior point x of A. Given $\varepsilon > 0$, there is a positive δ such that $\left|\Phi'(x)|B| - \Phi(B)\right| < \varepsilon|B|$ for each figure B with $d(B \cup \{x\}) < \delta$ and $r(B, x) > \varepsilon$.*

Lemma 2 (See [**BGP**, Proposition 3.3].) *Let f be integrable on A in one of the senses described in Definition 1. Given $\varepsilon > 0$, there is a gage δ in A such* *that* $\sum_{h=1}^{p} \left| f(x_h)|A_h| - \int_{A_h} f\,dx \right| < \varepsilon$ *for each* δ*-fine partition* $\{(A_h, x_h) : h = 1, \ldots, p\}$ *in* A *which is* ε*-regular or* ε*-regular special or* ε*-regular tight or* ε*-regular special and tight according to whether the integral is interpreted as* R_o^* *or* R_s^* *or* R_t^* *or* R_{st}^**, respectively.*

Lemma 3 *Let* F *be a continuous function on* A *and suppose for a given* $\varepsilon > 0$*, a given gage* δ *and a given set of points* $\{x_1, \ldots, x_p\}$ *the inequality*

$$\sum_h |F(B_h)| < \varepsilon, \tag{2}$$

holds for each ε*-regular* δ*-fine tight partition* $\{(B_h, x_h)\}$ *in* A*. Let* $\{(A_h, x_h) : h = 1, \ldots, p\}$ *be a* δ*-fine tight partition in* A *with the same set of points* $\{x_1, \ldots, x_p\}$*. Then, if for each* $h = 1, \ldots, p$ *the figure* A_h^r *is a subfigure of* A_h *which is* 4ε*-regular with respect to* x_h*, we have* $\sum_h |F(A_h^r)| < 2\varepsilon$ *.*

PROOF. We define a sequence of new partitions in A. Note that it might be that x_h does not belong to A_h^r but it does belong to A_h. We put for each $n = 1, 2, \ldots$ and for each $h = 1, \ldots, p$

$$B_h^{(n)} = \begin{cases} A_h^r & \text{if } x_h \in A_h^r, \\ A_h^r \cup [a_h^{(n)}, b_h^{(n)}] & \text{otherwise} \end{cases}$$

where $x_h \in [a_h^{(n)}, b_h^{(n)}] \subset A_h$, $|F(b_h^{(n)}) - F(a_h^{(n)})| < \frac{1}{n2^{h+1}}$ and $b_h^{(n)} - a_h^{(n)} < d(A_h^r \cup \{x_h\})$. Then for each n we have $x_h \in B_h^n$, $d(B_h^{(n)} \cup \{x_h\}) \leq 2d(A_h^r \cup \{x_h\})$ and $\| B_h^{(n)} \| \leq 2 \| A_h^r \|$ and so

$$r(B_h^{(n)}, x_h) = \frac{|B_h^{(n)}|}{d(B_h^{(n)} \cup \{x_h\}) \, \| B_h^{(n)} \|} \geq$$

$$\frac{|A_h^r|}{4d(A_h^r \cup \{x_h\}) \, \| A_h^r \|} = \frac{r(A_h^r, x_h)}{4} > \varepsilon.$$

Thus for n fixed $\{(B_h^{(n)}, x_h)\}$ is an ε-regular δ-fine tight partition in A and we get from (2) that for all n we have $\sum_h \left| F(B_h^{(n)}) \right| < \varepsilon$. Since $F(B_h^{(n)}) = F(A_h^r) + F([a_h^{(n)}, b_h^{(n)}])$ or $F(B_h^{(n)}) = F(A_h^r)$, for all n we have the estimate

$$\sum_h |F(A_h^r)| \leq \sum_h |F(B_h^{(n)})| + \sum_h |F([a_h^{(n)}, b_h^{(n)}])| < \varepsilon + \frac{1}{n}.$$

Letting $n \to \infty$ we obtain $\sum_h |F(A_h^r)| \leq \varepsilon < 2\varepsilon$ as required.□

Theorem 1 *Let f be integrable on $A = [a, b]$ in one of the senses of Definition 1 and let g be a function of bounded variation on A. Then fg is integrable on A in the same sense.*

PROOF. The R^*_{st}-integral case is known as it is covered by formula (1). We are going to give a proof for the case of the R^*_t-integral. The proofs for the other cases can be obtained in a similar manner with some modifications which are indicated below.

Let $F(x) = (R^*_t) \int_a^x f(t)\,dt$. As R^*_t-integral is included in H-integral, we can use the integration by parts formula (1) for interval $[a, x]$:

$$(3) \qquad (H)\int_a^x fg\,dt = [Fg]_a^x - (L)\int_a^x F\,dg.$$

Define $\Phi(x) = (H)\int_a^x fg\,dt$, $E = \{x \in A : \Phi'(x) = f(x)g(x)\}$ and $N = [a, b] \setminus E$. It is clear that $|N| = 0$. Without loss of generality we can suppose that $f(x) = 0$ for each $x \in N$ and $g(x)$ is increasing and positive on A.

Now fix $\varepsilon > 0$. For each $x \in E$ we can apply Lemma 1 to find $\delta_1(x) > 0$ such that the inequality

$$(4) \qquad |f(x)g(x)|B| - \Phi(B)| < \frac{\varepsilon|B|}{3(|A|+1)}$$

holds for each figure B with $d(B \cup \{x\}) < \delta_1(x)$ and $r(B, x) > \varepsilon/3(|A| + 1)$.

Next we are to define a gage for $x \in N$. We apply Lemma 2 to find $\delta_2(x) > 0$ for function f and for

$$(5) \qquad \varepsilon_1 = \frac{\varepsilon}{16(||g||_\infty + 1)}$$

instead of ε, where $||g||_\infty$ stands for the sup-norm. We also choose $\sigma > 0$ so that

$$(6) \qquad |F(x) - F(y)| < \frac{\varepsilon}{6(||g||_\infty + 1)} \text{ if } x, y \in A \text{ and } |x - y| < \sigma .$$

(We are using the fact that F is uniformly continuous on $[a, b]$.) Now we put

$$(7) \qquad \delta(x) = \begin{cases} \delta_1(x) & \text{if} \quad x \in E \\ \min(\delta_2(x), \sigma) & \text{if} \quad x \in N \end{cases} .$$

Having chosen a gage δ let $\{(A_h, x_h)\}$ be an ε-regular δ-fine tight partition of A. Now to prove the statement of the theorem we estimate the sum

$$(8) \qquad \begin{aligned} &|\textstyle\sum_h f(x_h)g(x_h)|A_h| - \Phi(A)| \le \\ &\textstyle\sum_h |f(x_h)g(x_h)|A_h| - \Phi(A_h)| \le \sum_{h\in I} + \sum_{h\in J}, \end{aligned}$$

where I is the set of all those indices h for which $x_h \in E$ and J is the set of all those h for which $x_h \in N$.

Taking into account that $r(A_h, x_h) > \varepsilon > \varepsilon/3(|A|+1)$ and applying (4) to A_h, $h \in I$, we get

$$\sum_{h\in I} < \frac{\varepsilon}{3}\,. \tag{9}$$

To estimate $\sum_{h\in J}$ note that $f(x_h) = 0$ for $x_h \in N$. Using (3) and putting $A_h = \cup_j[\alpha_j^h, \beta_j^h]$ we compute

$$\begin{aligned}
\sum_{h\in J} = \sum_{h\in J} |\Phi(A_h)| \\
&= \sum_{h\in J}\left|\sum_j \left(F(\beta_j^h)g(\beta_j^h) - F(\alpha_j^h)g(\alpha_j^h) - \int_{\alpha_j^h}^{\beta_j^h} F\,dg\right)\right| \\
&= \sum_{h\in J}\Big|\sum_j \left(F(\beta_j^h) - F(\alpha_j^h)\right) g(\beta_j^h) \\
&\quad + F(\alpha_j^h)\left(g(\beta_j^h) - g(\alpha_j^h)\right) - F(\xi_j)\left(g(\beta_j^h) - g(\alpha_j^h)\right)\Big| \\
&\le \sum_{h\in J}\left|\sum_j \left(F(\beta_j^h) - F(\alpha_j^h)\right) g(\beta_j^h)\right| \\
&\quad + \sum_{h\in J}\sum_j \left|F(\alpha_j^h) - F(\xi_j)\right| \left(g(\beta_j^h) - g(\alpha_j^h)\right) \\
&= S_1 + S_2
\end{aligned} \tag{10}$$

where we have applied the mean value theorem to choose $\xi_j \in [\alpha_j^h, \beta_j^h]$. From (6) and (7) we conclude that

$$S_2 \le \frac{\varepsilon}{6(\|g\|_\infty + 1)} 2\|g\|_\infty \le \frac{\varepsilon}{3}. \tag{11}$$

Now for each index $h \in J$ denote by A_h^+ a subfigure of A_h which is the union of all those connected components of A_h on which the increments of F are positive and by A_h^- the complementary subfigure of A_h. Now $|A_h| = |A_h^+| + |A_h^-|$ and one of these two subfigures has measure equal or greater than $|A_h|/2$. Denote this figure by A_h^r and the complementary subfigure by A_h^c. Since $r(A_h, x_h) > \varepsilon$, it is easy to check that $r(A_h^r, x_h) > \frac{\varepsilon}{2}$. Then (A_h, x_h) is also ε_1-regular and (A^r_h, x_h) is $4\varepsilon_1$-regular (see (5)). Noting once again that $f(x_h) = 0$ for $h \in J$ we can apply Lemma 2 to get

$$\sum_{h\in J} |F(A_h)| < \varepsilon_1 \tag{12}$$

and Lemma 3 to get

$$\sum_{h\in J} |F(A^r_h)| < 2\varepsilon_1 \ . \tag{13}$$

Since $F(A_h) = F(A^r_h) + F(A^c_h)$, we have $|F(A^c_h)| \le |F(A_h)| + |F(A^r_h)|$. Then from (12) and (13) we get

$$\sum_{h\in J} |F(A^c_h)| \le \sum_{h\in J} |F(A_h)| + \sum_{h\in J} |F(A^r_h)| < 3\varepsilon_1 \ . \tag{14}$$

It also follows from the definitions of A^r_h and A^c_h that for each $h \in J$

$$\left| \sum_j \left(F(\beta^h_j) - F(\alpha^h_j)\right) g(\beta^h_j) \right| \le \left(|F(A^r_h)| + |F(A^c_h)|\right) \|g\|_\infty \ .$$

Therefore by (13), (14) and (5) we get

$$S_1 \le 5\varepsilon_1 \|g\|_\infty < \frac{\varepsilon}{3} \ . \tag{15}$$

Finally summing up the inequalities (9), (11) and (15) and taking (8) and (10) into account we obtain the estimate $|\sum_h f(x_h)g(x_h)|A_h| - \Phi(A)| < \varepsilon$ for any ε-regular δ-fine tight partition of A. Thus we have proved that fg is R^*_t-integrable on A and that $\Phi(A)$ is the R^*_t-integral of fg on A.

The cases of the R^*_o-integral and the R^*_s-integral are in fact simpler. For the R^*_o-integral we don't need Lemma 3 to get (13) as it follows directly from Lemma 2 applied to $\{(A^r_h, x_h)\}$.

In the case of the R^*_s-integral we have just one member in the inner sum of S_1 in (10), and so there is no need to split the figure A_h into two subfigures A^+_h and A^-_h to get the desirable estimate. □

Note that we have also proved that for R^*_o- or R^*_s- or R^*_t-integrable function f formula (1) holds if we replace H-integral by the corresponding R^*-integral.

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