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# APPROXIMATE CORE TOPOLOGIES

## 1 Introduction

The purpose of this paper is to introduce some topologies connected with the notions of density and $\mathcal{I}$-density in ways which are analogous to the core topology on the plane. The core topology can be considered in every linear space $X$ and is related to the linear structure of the space. It is an example (important in the theory of functional equations) of a semilinear topology which is not linear if the dimension of $X$ is greater than one (cf. [K], [KK], [Ko], [V]). Topologies presented here are modifications of the core topology similar to the $d$-crosswise topology (cf. [LMZ], page 98.)

Recall the basic notions.

Let $A \subset \mathbb{R}^2$. A point $x \in A$ is said to be an algebraically interior point of $A$ if and only if for each $y \in \mathbb{R}^2$ there exists $\varepsilon > 0$ such that $x + ty \in A$ for $t \in (-\varepsilon, \varepsilon)$, i.e. $A$ contains an open segment centered at $x$ in every direction. The set of all points which are algebraically interior to $A$ is denoted by core $A$. A set $A \subset \mathbb{R}^2$ is called algebraically open if $A = \operatorname{core} A$. The family

$$\mathcal{T} = \{A \subset \mathbb{R}^2 : A = \operatorname{core} A\}$$

forms a topology in $\mathbb{R}^2$ called the core topology.

The paper consists of two parts. In the first one we introduce two topologies: $\mathcal{T}'_{apc}$ and $\mathcal{T}_{apc}$ analogous to the core topology on the plane. Here the role of the Euclidean topology on the real line in every direction is played by the density topology $d_1$ on the real line. We demonstrate that $\mathcal{T}_{apc}$ is stronger than $d_1 \times d_1$ and weaker than the ordinary density topology $d_2$ on the plane; the topologies $\mathcal{T}_{apc}$ and $d_2^s$ (strong density topology on the plane) are incomparable. For the terminology and definitions of density point and the density topologies on the real line and on the plane, see [GNN].

In the second part, analogous considerations are carried out for Baire category, but different results are obtained. The topology $\mathcal{T}_{\mathcal{I}apc}$ is stronger than

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the product of $\mathcal{I}$-density topologies $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1}$, but the topologies $\mathcal{T}_{\mathcal{I}apc}$ and $d_{\mathcal{I}_2}$, and also $\mathcal{T}_{\mathcal{I}apc}$ and $d^s_{\mathcal{I}_2}$, are incomparable, where $d_{\mathcal{I}_2}$, $d^s_{\mathcal{I}_2}$ denote the $\mathcal{I}$-density topology and the strong $\mathcal{I}$-density topology on the plane, respectively. For the terminology and definitions of $\mathcal{I}$-density point and the $\mathcal{I}$-density topologies on the real line and on the plane, see [PWW1], [PWW2], [W] and [CW].

## 2 The Measure Density Case

Let $\mathbb{R}$ denote the real line, $\mathbb{R}^2$, the plane and $\mathbb{N}$, the set of positive integers.

The ball centered at a point $(x, y)$ and with radius $r$ will be denoted by $K((x, y), r)$.

We denote by $m_k$ ($m_k^*$) Lebesgue measure (outer Lebesgue measure) on $\mathbb{R}^k$, $k = 1, 2$.

Let $d_k(E, p)$ denote the density of the measurable set $E$ at a point $p$ in $\mathbb{R}^k$, $k = 1, 2$.

If a plane set is contained in a line, then we shall consider its linear measure and we use the linear density ($d_1$).

It will be necessary to consider various topological operations such as closure (Cl) or interior (Int), with respect to several different topologies. For this reason, we have adopted the convention of preceding any such notion by the symbol of the topology. When no prefix appears, it should be assumed that the Euclidean topology is meant.

Let $A \triangle B$ denote the symmetric difference of $A$ and $B$.

**Definition 2.1** *We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is an approximate core interior point or simply an apc-interior point of a set $A \subset \mathbb{R}^2$ if and only if $(x_0, y_0) \in A$ and, for each line $p$ passing through the point $(x_0, y_0)$, the inner density of $A \cap p$ at $(x_0, y_0)$ is 1, that is there exists a linearly measurable set $B \subset A \cap p$ such that*

$$d_1(B, (x_0, y_0)) = 1. \tag{1}$$

The set of all apc-interior points of $A \subset \mathbb{R}^2$ will be called the approximate core of $A$ and denoted by $ap$–core $A$:

$$ap\text{–core}\, A = \{a \in A : a \text{ is an apc-interior point of } A\}.$$

Put $\mathcal{T}'_{apc} = \{A \subset \mathbb{R}^2 : A = ap\text{–core}\, A\}$. Since (1) is preserved under finite intersections and arbitrary unions, we obtain that $\mathcal{T}'_{apc}$ is a topology on the plane, which we may think of as the unrestricted approximate core topology.

Obviously, $\mathcal{T}'_{apc}$- Int $A \subset ap$–core $A$ for each $A \subset \mathbb{R}^2$. The inverse inclusion need not hold. Indeed, let $E = \bigcup_{n=1}^{\infty}(a_n, b_n) \cup \{0\} \cup \bigcup_{n=1}^{\infty}(-b_n, -a_n)$ where

$a_{n+1} < b_{n+1} < a_n$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} b_n = 0$ and $d_1(E, 0) = 1$. Put

$$A = K((1,0),1) \cup K((-1,0),1) \cup (\{0\} \times E).$$

Then $(0,0) \in ap$–core $A$, but $(0,0) \notin \mathcal{T}'_{apc}$-Int $A$.

It would be easier to take the interval $[-1,1]$ instead of the more complicated set $E$, but we obtained simultaneously an example of a set $A$, for which $(0,0)$ belongs to ap-core $A$ and does not belong to core $A$.

If $E \subset \mathbb{R}^2$ and if $a \in \mathbb{R}$, then let

$$(a,a)E = \{(ax, ay) : (x,y) \in E\} = \{(\Theta, ar) : (\Theta, r) \in E\}$$

(the last description in polar co-ordinates), and let $(E)_\alpha$ denote the set $E$ rotated by an angle $\alpha$.

Let $d_1 \times d_1$ denote the product of two density topologies.

**Theorem 2.1** $d_1 \times d_1 \subsetneqq \mathcal{T}'_{apc}$.

PROOF. Let $E \subset \mathbb{R}^2$, $E \in d_1 \times d_1$ and let $(x_0, y_0)$ be an arbitrary point of $E$. We may assume that $x_0 = y_0 = 0$. There exist two $d_1$-open sets $A$, $B$ such that $0 \in A$, $0 \in B$ and $A \times B \subset E$. We shall show that $(0,0)$ is an *apc*-interior point of $E$. Let $p$ be an arbitrary line passing through $(0,0)$, $p = \{r\cos\alpha, r\sin\alpha),\ r \in \mathbb{R}\}$. Since the desired conclusion is clear if $p$ lies on coordinate axes, we may assume that $\cos\alpha \neq 0$ and $\sin\alpha \neq 0$. If

$$C_\alpha = \frac{1}{\cos\alpha} A \cap \frac{1}{\sin\alpha} B,$$

then $d_1(C_\alpha, 0) = 1$, and clearly $\{(r\cos\alpha, r\sin\alpha) : r \in C_\alpha\} \subset p \cap E$, so $d_1(p \cap E, (0,0)) = 1$. Consequently, $(x_0, y_0)$ is an *apc*-interior point of $E$, and $E \in \mathcal{T}'_{apc}$.

Observe that $d_1 \times d_1 \neq \mathcal{T}'_{apc}$ because $\mathcal{T}'_{apc}$ is rotation invariant and $d_1 \times d_1$ does not have this property (see [WB]).□

It is easy to see that in the topology $\mathcal{T}'_{apc}$ there are also non-measurable sets. Sierpiński proved (Fund. Math. vol. 1, page 112, see also [O], th. 14.4) that there exists a plane set $E$ such that $E$ meets every closed set of positive plane measure, and no three points of $E$ are collinear. It is clear that such a set $E$ cannot be measurable. Obviously, $\mathbb{R}^2 \setminus E$ is also non-measurable, and $\mathbb{R}^2 \setminus E \in \mathcal{T}'_{apc}$. However, if $A \in \mathcal{T}'_{apc}$ and $p$ is a straight line, then $A \cap p$ is measurable (as a linear set).

**Definition 2.2** *We say that a set $A \subset \mathbb{R}^2$ is approximately core open or simply apc-open if and only if $A$ is a measurable set and each point $(x,y) \in A$ is an apc-interior point of $A$.*

Let $\mathcal{L}_2$ denote the family of all Lebesgue measurable subsets of the plane. Put $\mathcal{T}_{apc} = \{A \in \mathcal{L}_2 : A = ap\text{-core}\, A\}$ Obviously, the family $\mathcal{T}_{apc}$ is identical with the family of all *apc*-open sets, and $\mathcal{T}_{apc} = \mathcal{T}'_{apc} \cap \mathcal{L}_2$. We shall show that $\mathcal{T}_{apc}$ is a topology on the plane, stronger than $d_1 \times d_1$ and weaker than the ordinary density topology $d_2$ on the plane. We may call $\mathcal{T}_{apc}$ the approximate core topology on the plane.

Let $p_\Theta$ denote the half-line running from the origin and forming an angle $\Theta$ with the $x$-axis, $\Theta \in [0, 2\pi)$. We shall need the following lemmas:

**Lemma 2.2** *Let $H \subset [0, 2\pi)$ be a measurable set and let $0 < \alpha < r$. If $E \subset K((0,0), r)$ is a plane measurable set ($E \in \mathcal{L}_2$) such that*

$1^0$ *$E_\Theta = E \cap p_\Theta$ is linearly measurable for each $\Theta \in H$,*

$2^0$ *$m_1(E_\Theta) < \alpha$ for each $\Theta \in H$,*

*then*

$$\int_H \left( \int_0^r \chi_{E_\Theta}(t) t\, dt \right) d\Theta \le \frac{r^2 - (r-\alpha)^2}{2} m_1(H).$$

PROOF. Let $\Theta \in H$. Observe that

$$\int_0^r \chi_{E_\Theta}(t) t\, dt \le \int_{r-\alpha}^r t\, dt = \frac{r^2 - (r-\alpha)^2}{2}.$$

Consequently, $\int_H \left( \int_0^r \chi_{E_\Theta}(t) t\, dt \right) d\Theta \le \frac{r^2 - (r-\alpha)^2}{2} m_1(H)$. So, the proof of Lemma is accomplished. □

Let $p_\Theta(h)$ denote the interval on the half-line $p_\Theta$, with endpoint at the origin and with length $h$.

**Lemma 2.3** *If $G \subset K((0,0), r)$ is open, then the set*

$$\{\Theta \in [0, 2\pi) : m_1(G \cap p_\Theta(r)) > r'\}$$

*is open for each $r' < r$.*

PROOF. Let $\Theta_0 \in \{\Theta \in [0, 2\pi) : m_1(G \cap p_\Theta(r)) > r'\}$. The set $G \cap p_{\Theta_0}(r)$ is open on the line $p_{\Theta_0}$, so it can be represented as the union of a sequence of nonoverlapping intervals $(a_n, b_n)$, $n \in \mathbb{N}$. Obviously, $m_1\left( \bigcup_{n=1}^{\infty} (a_n, b_n) \right) > r'$. Determine $n_0$ so that $m_1\left( \bigcup_{n=1}^{n_0} (a_n, b_n) \right) > r'$ and let $[c_n, d_n]$ be a closed interval such that $[c_n, d_n] \subset (a_n, b_n)$ for $n = 1, \ldots, n_0$ and $m_1\left( \bigcup_{n=1}^{n_0} [c_n, d_n] \right) > r'$.

For each point $p \in \bigcup_{n=1}^{n_0}[c_n, d_n]$ there exists an open set $V_p$ (in the polar co-ordinates) of the form

$$V_p = \{(\Theta, r) : \Theta \in (\Theta_0 - \Theta_p, \Theta_0 + \Theta_p),\ r \in (\rho((0,0),p) - r_p, \rho((0,0),p) + r_p)\},$$

where $\Theta_p > 0$, $r_p > 0$, such that $p \in V_p$, $V_p \subset G$ and $\rho$ denotes the Euclidean distance in $\mathbb{R}^2$. The family of sets $\{V_p \cap p_{\Theta_0},\ p \in \bigcup_{n=1}^{n_0}[c_n, d_n]\}$ is an open covering of $\bigcup_{n=1}^{n_0}[c_n, d_n]$. By the Borel-Lebesgue theorem there exists a finite subfamily such that $\bigcup_{n=1}^{n_0}[c_n, d_n] \subset \bigcup_{i=1}^{k} V_{p_i} \cap p_{\Theta_0}$. Put $\alpha = \min\{\Theta_{p_1}, \ldots, \Theta_{p_k}\}$. Then $m_1(G \cap p_\Theta(r)) \geq m_1\Big(\bigcup_{n=1}^{n_0}[c_n, d_n]\Big) > r'$ for $\Theta \in (\Theta_0 - \alpha, \Theta_0 + \alpha)$, which completes the proof.□

**Theorem 2.4** *$\mathcal{T}_{apc}$ is a topology on the plane, stronger than $d_1 \times d_1$ and weaker than the ordinary density topology $d_2$ on the plane.*

PROOF. The only difficulty in this proof is to show that the union of an arbitrary subfamily of $\mathcal{T}_{apc}$ is a measurable set on the plane. For this purpose, it is sufficient to demonstrate that $\mathcal{T}_{apc} \subset d_2$. Let $A \in \mathcal{T}_{apc}$ and $(x_0, y_0) \in A$. We may assume that $x_0 = y_0 = 0$. Let $\varepsilon > 0$. For each $\Theta \in [0, 2\pi)$, there exists $r_\Theta > 0$ such that if $0 < h < r_\Theta$, then $m_1(A \cap p_\Theta(h)) > h(1 - \varepsilon/4)$. (Recall that all sets $A \cap p_\Theta(h)$ are measurable.)

Suppose that we have chosen $r_\Theta$ with the above mentioned property for each $\Theta \in [0, 2\pi)$. Put $B_n = \{\Theta \in [0, 2\pi) : r_\Theta > 1/n\}$. We have $B_n \subset B_{n+1}$ for every $n$, and $\bigcup_{n=1}^{\infty} B_n = [0, 2\pi)$. Thus $m_1^*(B_n) \xrightarrow[n\to\infty]{} 2\pi$. There exists $n_0 \in \mathbb{N}$ such that $m_1^*(B_{n_0}) > 2\pi - \varepsilon$. Consequently, if $\Theta \in B_{n_0}$ and $0 < h < 1/n_0$, then $m_1(A \cap p_\Theta(h)) > h(1 - \varepsilon/4)$.

Let $0 < h < 1/n_0$. Obviously, $A \cap K((0,0), h) \supset A \cap \Big(\bigcup_{\Theta \in B_{n_0}} p_\Theta(h)\Big)$. We shall show that $m_2^*\Big(A \cap \bigcup_{\Theta \in B_{n_0}} p_\Theta(h)\Big) \geq \pi h^2(1 - \varepsilon)$. Let $G$ be an arbitrary open set such that $A \cap \bigcup_{\Theta \in B_{n_0}} p_\Theta(h) \subset G \subset K((0,0), h)$. For $\Theta \in B_{n_0}$, we have $A \cap p_\Theta(h) \subset G \cap p_\Theta(h)$, so $m_1(G \cap p_\Theta(h)) > h(1 - \varepsilon/4)$. Let us consider the set $H = \{\Theta \in [0, 2\pi) : m_1(G \cap p_\Theta(h)) > h(1 - \varepsilon/4)\}$. Obviously, $B_{n_0} \subset H$. From Lemma 2.3 it follows that $H$ is open. Clearly, $m_1(H) \geq m_1^*(B_{n_0}) > 2\pi - \varepsilon$. Put $E = K((0,0), h) \setminus G$. For $\Theta \in H$, we have $m_1(E_\Theta) = m_1(p_\Theta(h) \setminus (G \cap p_\Theta(h))) < h \cdot \varepsilon/4$, so, by Lemma 2.2,

$$\begin{aligned} m_2(E) &= \int_0^{2\pi} \Big(\int_0^h \chi_{E_\Theta}(t) t\, dt\Big) d\Theta = \int_H \Big(\int_0^h \chi_{E_\Theta}(t) t\, dt\Big) d\Theta \\ &\quad + \int_{[0,2\pi)\setminus H} \Big(\int_0^h \chi_{E_\Theta}(t) t\, dt\Big) d\Theta \leq \frac{h^2}{2}\big(1 - (1 - \frac{\varepsilon}{4})^2\big) \cdot 2\pi + \frac{h^2}{2} \cdot \varepsilon \\ &< \pi h^2 \varepsilon. \end{aligned}$$

Hence $m_2(G) > \pi h^2(1-\varepsilon)$ and, consequently, by the arbitrariness of $G$

$$m_2^*\Big(A \cap \bigcup_{\Theta \in B_{n_0}} p_\Theta(h)\Big) \geq \pi h^2(1-\varepsilon).$$

Therefore $m_2(A \cap K((0,0),h)) \geq \pi h^2(1-\varepsilon)$. Letting $\varepsilon \to 0$, we conclude that $d_2(A,(0,0)) = 1$.

We have thus shown that $\mathcal{T}_{apc} \subset d_2$. By Theorem 2.1 $d_1 \times d_1 \subset \mathcal{T}'_{apc}$. Simultaneously, $d_1 \times d_1 \subset \mathcal{L}_2$, so $d_1 \times d_1 \subset \mathcal{T}_{apc}$. Evidently, $d_1 \times d_1 \neq \mathcal{T}_{apc}$ because $\mathcal{T}_{apc}$ is rotation invariant and $d_1 \times d_1$ does not have this property. We shall prove that $\mathcal{T}_{apc} \neq d_2$. Put $E = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : 0 < x,\ -x^2 \leq y \leq x^2\}$. Then $E \in d_2 \setminus \mathcal{T}_{apc}$ because $d_1(E \cap p_0,(0,0)) = \frac{1}{2}$.□

We show that the topologies $\mathcal{T}_{apc}$ and $d_2^s$ are incomparable.

**Theorem 2.5** $d_2^s \not\subset \mathcal{T}_{apc}$, $\mathcal{T}_{apc} \not\subset d_2^s$.

PROOF. Put $A = (\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : y = 0\}) \cup \{(0,0)\}$. It is easy to see that $A \in d_2^s \setminus \mathcal{T}_{apc}$ because $d_1(A \cap p_0,(0,0)) = 0$.

Now, let $E = \bigcup_{n=1}^{\infty}(a_n,b_n)$ where $a_{n+1} < b_{n+1} < a_n$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} b_n = 0$ and $d_1((-\infty,0] \cup E, 0) = 1$. Let $G_n$ be an open set on the plane such that $(a_n,b_n) \times \{0\} \subset G_n \subset \{(x,y) \in \mathbb{R}^2 : 0 < x,\ -x^4 < y < x^4\}$. Put

$$B = (\mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : 0 < x,\ -x^2 \leq y \leq x^2\}) \cup \bigcup_{n=1}^{\infty} G_n.$$

Clearly, $B \in \mathcal{T}_{apc}$. On the other hand,

$$\frac{m_2((\mathbb{R}^2 \setminus B) \cap [0,1/n) \times [0,1/n^4))}{m_2([0,1/n) \times [0,1/n^4))} \geq \frac{4}{5} - \frac{2}{3n} \xrightarrow[n\to\infty]{} \frac{4}{5} > 0,$$

so the upper strong density of $\mathbb{R}^2 \setminus B$ at $(0,0)$ is positive. Consequently, $B \notin d_2^s$.□

It is easy to see that the core topology is weaker than $\mathcal{T}_{apc}$ and the set $B$ from Theorem 2.5 shows that, in fact, the last is essentially stronger.

## 3 The Category Density Case

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ denote the families of meager sets on the real line and on the plane, respectively. If a plane set is contained in a real line, then we shall consider its linear $\mathcal{I}$-density $(d_{\mathcal{I}_1})$.

**Definition 3.1** *We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is an $\mathcal{I}$-approximate core interior point or more simply an $\mathcal{I}apc$-interior point of a set $A \subset \mathbb{R}^2$ if and only if $(x_0, y_0) \in A$ and, for each line $p$ passing through the point $(x_0, y_0)$, the inner $\mathcal{I}$-density of $A \cap p$ at $(x_0, y_0)$ is 1, that is there exists a set $B$ having the Baire property such that $B \subset A \cap p$ and*

$$d_{\mathcal{I}_1}(B, (x_0, y_0)) = 1. \tag{2}$$

The set of all $\mathcal{I}apc$-interior points of $A$ will be called $\mathcal{I}$-approximate core of $A$ and denoted by $\mathcal{I}\ ap\text{–core}\, A$:

$$\mathcal{I}\ ap\text{–core}\, A = \{a \in A : a \text{ is an } \mathcal{I}apc\text{-interior point of } A\}.$$

Put $\mathcal{T}'_{\mathcal{I}apc} = \{A \subset \mathbb{R}^2 : A = \mathcal{I}ap\text{-core } A\}$. Since (2) is preserved under finite intersections and arbitrary unions, we obtain that $\mathcal{T}'_{\mathcal{I}apc}$ is a topology on the plane, which we may think of as the unrestricted $\mathcal{I}$-approximate core topology on the plane.

Let $d_{\mathcal{I}_1}$ denote the $\mathcal{I}$-density topology on the real line (see [PWW1], [PWW2]) and let $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1}$ denote the product of two $\mathcal{I}$-density topologies.

**Theorem 3.1** $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1} \subsetneq \mathcal{T}'_{\mathcal{I}apc}$.

The proof of this theorem is analogous to that of Theorem 2.1.□

It is easy to see that the topology $\mathcal{T}'_{\mathcal{I}apc}$ contains some sets without the Baire property. There exists a plane set $E$ of the second category such that no three points of $E$ are collinear (see [O], th. 15.5). It is clear that such a set $E$ cannot have the Baire property. Obviously, $\mathbb{R}^2 \backslash E$ lacks the property of Baire and $\mathbb{R}^2 \backslash E \in \mathcal{T}'_{\mathcal{I}apc}$. As in the case of measure, if $A \in \mathcal{T}'_{\mathcal{I}apc}$ and $p$ is a straight line, then $A \cap p$ has the Baire property (as a linear set).

**Definition 3.2** *We say that a set $A \subset \mathbb{R}^2$ is $\mathcal{I}$-approximate core open or simply $\mathcal{I}apc$-open if and only if $A$ has the Baire property and every point $(x_0, y_0) \in A$ is an $\mathcal{I}apc$-interior point of $A$.*

Let $\mathcal{B}_2$ denote the family of all sets having the Baire property on the plane. Put $\mathcal{T}_{\mathcal{I}apc} = \{A \in \mathcal{B}_2 : A = \mathcal{I}ap\text{-core } A\}$. Obviously, the family $\mathcal{T}_{\mathcal{I}apc}$ is identical with the family of all $\mathcal{I}apc$-open sets, and $\mathcal{T}_{\mathcal{I}apc} = \mathcal{T}'_{\mathcal{I}apc} \cap \mathcal{B}_2$. We show that $\mathcal{T}_{\mathcal{I}apc}$ is a topology on the plane, stronger than $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1}$, but incomparable with ordinary $\mathcal{I}$-density topology $d_{\mathcal{I}_2}$ on the plane (for the definition and properties of $d_{\mathcal{I}_2}$ see [W], [CW]). We may call $\mathcal{T}_{\mathcal{I}apc}$ the $\mathcal{I}$-approximate core topology on the plane.

**Lemma 3.2** *If $A \in \mathcal{T}_{\mathcal{I}apc}$ and $A = (G \backslash P_1) \cup P_2$, where $G$ is an open set and $P_1, P_2 \in \mathcal{I}_2$, then $P_2 \subset \mathrm{Cl}\, G$.*

PROOF. Let $(x,y) \in P_2$ and suppose that $(x,y) \notin \mathrm{Cl}\, G$. We may assume that $x = y = 0$. Then there exists $r > 0$ such that $K((0,0),r) \cap A \in \mathcal{I}_2$. From the Kuratowski-Ulam theorem for polar co-ordinates, the set $K((0,0),r) \cap A \cap p_\Theta$ is of the first category (on the half-line $p_\Theta$) for all $\Theta \in [0,2\pi)$ except a set of the first category. Simultaneously, $(0,0) \in A$ and $A \in \mathcal{T}_{\mathcal{I}apc}$. Consequently, $(0,0)$ is an $\mathcal{I}apc$-interior point of $A$ - a contradiction.□

**Theorem 3.3** *$\mathcal{T}_{\mathcal{I}apc}$ is a topology on the plane, stronger than $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1}$.*

PROOF. The only difficulty in this proof is to show that the union of an arbitrary subfamily of $\mathcal{T}_{\mathcal{I}apc}$ is a set having the Baire property on the plane. Let $\mathcal{A} = \{A_\alpha,\ \alpha \in \Gamma\} \subset \mathcal{T}_{\mathcal{I}apc}$. Then $A_\alpha = (G_\alpha \backslash P_{1\alpha}) \cup P_{2\alpha}$, where $G_\alpha$ is an open set and $P_{1\alpha}, P_{2\alpha} \in \mathcal{I}_2$ for $\alpha \in \Gamma$. By the theorem of Lindelöf there exists the sequence $\{\alpha_n\}_{n\in\mathbb{N}} \subset \Gamma$ such that $\bigcup_{\alpha\in\Gamma} G_\alpha = \bigcup_{n=1}^{\infty} G_{\alpha_n}$. From Lemma 3.2 we have

$$\begin{aligned}\bigcup_{n=1}^{\infty} G_{\alpha_n} \backslash \bigcup_{n=1}^{\infty} P_{1\alpha_n} \subset \bigcup_{n=1}^{\infty} (G_{\alpha_n} \backslash P_{1\alpha_n}) \subset \bigcup_{\alpha\in\Gamma} [(G_\alpha \backslash P_{1\alpha}) \cup P_{2\alpha}] \\ \subset \bigcup_{\alpha\in\Gamma} \mathrm{Cl}\, G_\alpha \subset \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} G_{\alpha_n}\right) = \bigcup_{n=1}^{\infty} G_{\alpha_n} \cup P_0,\end{aligned}$$

where $P_0 = \mathrm{Fr}\left(\bigcup_{n=1}^{\infty} G_{\alpha_n}\right)$. Obviously, $P_0 \in \mathcal{I}_2$, so $\bigcup_{\alpha\in\Gamma} A_\alpha \triangle \bigcup_{n=1}^{\infty} G_{\alpha_n} \in \mathcal{I}_2$ and $\bigcup_{\alpha\in\Gamma} A_\alpha \in \mathcal{B}_2$.

By Theorem 3.1, $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1} \subset \mathcal{T}'_{\mathcal{I}apc}$. Simultaneously, $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1} \subset \mathcal{B}_2$, so $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1} \subset \mathcal{T}_{\mathcal{I}apc}$. Evidently, $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1} \neq \mathcal{T}_{\mathcal{I}apc}$, because $\mathcal{T}_{\mathcal{I}apc}$ is rotation invariant and $d_{\mathcal{I}_1} \times d_{\mathcal{I}_1}$ does not have this property (see [WB]).□

But here ends the analogy between the properties of $\mathcal{T}_{apc}$ and $\mathcal{T}_{\mathcal{I}apc}$, between the measure and category. We show that the topologies $\mathcal{T}_{\mathcal{I}apc}$ and $d_{\mathcal{I}_2}$, where $d_{\mathcal{I}_2}$ is an $\mathcal{I}$-density topology on the plane (cf. [W], [CW]), are incomparable.

**Theorem 3.4** *There exists a set $A \subset \mathbb{R}^2$ having the Baire property on the plane such that*

$1^0$ *$(0,0)$ is an $\mathcal{I}_1$-dispersion point of $A \cap p_\Theta$ for each $\Theta \in [0,2\pi)$,*

$2^0$ *$(0,0)$ is not an $\mathcal{I}_2$-dispersion point of $A$.*

PROOF. Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers tending to 0 such that $a_{n+1}/a_n \xrightarrow[n\to\infty]{} 0$. Fix $n \in \mathbb{N}$. If $j \in \{0,1,2,\ldots,n^2-1\}$.

Then $j$ can be uniquely represented in the form $j = n \cdot k_j + i_j$, where $k_j, i_j \in \{0, 1, 2, \ldots, n-1\}$. Put

$$A_{n,j} = \{(\Theta, r) : (n \cdot i_j + k_j)\frac{\pi}{n^2} \le \Theta < (n \cdot i_j + k_j + 1)\frac{\pi}{n^2}, \\ a_n \cdot 2^{k_j/n} \le r < a_n \cdot 2^{(k_j+1)/n}\}$$

for $n \in \mathbb{N}$ and $j \in \{0, 1, 2, \ldots, n^2-1\}$ set $A_n = \bigcup_{j=0}^{n^2-1} A_{n,j}$ and $A = \bigcup_{n=1}^{\infty} A_n$. To prove $1^0$ fix $\Theta \in [0, \pi)$. For each $n \in \mathbb{N}$ there exists exactly one $j \in \{0, 1, 2, \ldots, n^2-1\}$ such that $p_\Theta$ intersects $A_{n,j}$. Thus $A \cap p_\Theta$ treated as a linear set is of the form $\bigcup_{n=1}^{\infty}[c_n, d_n)$, where $[c_n, d_n) \subset [a_n, 2a_n]$ and $d_n = c_n \cdot 2^{1/n}$. Hence $d_{n+1}/c_n \xrightarrow[n\to\infty]{} 0$, because $d_{n+1}/c_n \le 2a_{n+1}/a_n$, $(d_n - c_n)/c_n \xrightarrow[n\to\infty]{} 0$ and it suffices to use Theorem 2 from [W]. If $\Theta \in [\pi, 2\pi)$, then $A \cap p_\Theta = \emptyset$.

To prove $2^0$ take a sequence $\{t_n\}_{n\in\mathbb{N}}$ defined by $t_n = 1/(2a_n)$. Observe that if $(\Theta_0, r_0) \in P = \{(\Theta, r) : 0 \le \Theta \le \pi,\ 1/2 \le r \le 1\}$, then $\operatorname{dist}((t_n, t_n) \cdot A, (\Theta_0, r_0)) = \operatorname{dist}((t_n, t_n) \cdot A_n, (\Theta_0, r_0)) \le \pi/n$. Hence for arbitrary subsequence $\{n_m\}_{m\in\mathbb{N}}$ and for arbitrary $p \in \mathbb{N}$ we obtain that $\bigcup_{m=p}^{\infty}((t_{n_m}, t_{n_m}) \cdot A)$ includes an open set dense in $P$. Consequently

$$\limsup_m((t_{n_m}, t_{n_m}) \cdot A) \cap K((0,0), 1) \notin \mathcal{I}_2,$$

which completes the proof.□

**Corollary 3.5** *The topologies $\mathcal{T}_{\mathcal{I}apc}$ and $d_{\mathcal{I}_2}$ are incomparable.*

PROOF. Put $E = (\mathbb{R}^2 \setminus \operatorname{Cl} A) \cup \{(0,0)\}$, where $A$ is the set from the last theorem. Obviously, $E \in \mathcal{B}^2$. If $(x,y) \in E$ and $(x,y) \neq (0,0)$, then it is an interior point of $E$ in the Euclidean topology, so it is an $\mathcal{I}apc$-interior point of $E$. From the last theorem $(0,0)$ is an $\mathcal{I}_1$-dispersion point of $\operatorname{Cl} A \cap p_\Theta$ for each $\Theta \in [0, 2\pi)$. Hence $(0,0)$ is an $\mathcal{I}_1$-density point of $p_\Theta \setminus (\operatorname{Cl} A \cap p_\Theta) = (\mathbb{R}^2 \setminus \operatorname{Cl} A) \cap p_\Theta$. Consequently, $(0,0)$ is an $\mathcal{I}apc$-interior point of $E$ and $E \in \mathcal{T}_{\mathcal{I}apc}$. From the second condition in Theorem 3.4 it follows that $E \notin d_{\mathcal{I}_2}$. Now put $B = (\mathbb{R}^2 \setminus \{(x,y) : y = 0\}) \cup \{(0,0)\}$. It is easy to see that $B \in d_{\mathcal{I}_2} \setminus \mathcal{T}_{\mathcal{I}apc}$.□

Let $d^s_{\mathcal{I}_2}$ denote the strong $\mathcal{I}$-density topology on the plane (see [W], [CW]).

**Corollary 3.6** *The topologies $\mathcal{T}_{\mathcal{I}apc}$ and $d^s_{\mathcal{I}_2}$ are incomparable.*

The proof is analogous to the proof of Corollary 3.5.□

Obviously, the theorem analogous to Theorem 3.4 for measure does not hold. On the other hand, both for measure and category it is possible to construct a set $A$ which has a "plane" dispersion point in $(0,0)$, but $(0,0)$ is not a "linear" dispersion point of $A \cap p_\Theta$ for each $\Theta \in [0, \pi)$.

**Theorem 3.7** *There exists a set $A \subset \mathbb{R}^2$ which is measurable and has the Baire property such that*

$1^0$ $(0,0)$ *is a $d_2$- and $\mathcal{I}_2$-dispersion point of $A$,*

$2^0$ $(0,0)$ *is neither a $d_1$- nor a $\mathcal{I}_1$-dispersion point of $A \cap p_\Theta$ for each $\Theta \in [0,\pi)$.*

PROOF. Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers tending to 0 such that $a_{n+1}/a_n \xrightarrow[n\to\infty]{} 0$. Each natural number $n$ can be uniquely represented in the following way: $n = 2^{k_n} + i_n$, where $k_n$ is a natural number or zero and $i_n \in \{0,1,2,\ldots,2^{k_n}-1\}$. Put $\Theta_n = \pi \cdot i_n/2^{k_n}$ and

$$A_n = \{(\Theta,r) : \Theta_n \le \Theta < \Theta_n + \pi/2^{k_n},\ a_n \le r \le 2a_n\}.$$

Let $A = \bigcup_{n=1}^{\infty} A_n$. Obviously, $A$ is measurable and has the Baire property.

To prove $1^0$ take an arbitrary increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers tending to infinity. Let $j_n$ denote the natural number such that $a_{j_n} \le 1/t_n < a_{j_n-1}$. (In any case additionally put $a_0 = 1 + \max(a_1,\ 1/t_1)$.) Observe that

$$\begin{aligned}((t_n,t_n)\cdot A) \cap K((0,0),1) \subset \mathrm{Sec}(&\Theta_{j_n}, \Theta_{j_n} + \pi/2^{k_{j_n}})\\ &\cup K((0,0), 2a_{j_n+1}\cdot t_n),\end{aligned}$$

where $\mathrm{Sec}(\Theta',\Theta'') = \{(\Theta,r) : \Theta' \le \Theta \le \Theta'',\ r<1\}$.

Choose a subsequence $\{n_m\}_{m\in\mathbb{N}}$ such that $\{\Theta_{j_{n_m}}\}_{m\in\mathbb{N}}$ is convergent. Put $\Theta_0 = \lim_{m\to\infty}\Theta_{j_{n_m}}$. We show that $\limsup_m((t_{n_m},t_{n_m})\cdot A) \cap K((0,0),1) \subset p_{\Theta_0}$, which is obviously sufficient to show that $(0,0)$ is simultaneously a $d_2$- and $\mathcal{I}_2$-dispersion point of $A$.

Indeed, $\limsup_m((t_{n_m},t_{n_m})\cdot A) \cap K((0,0),1) \subset \limsup_m \mathrm{Sec}(\Theta_{j_{n_m}}, \Theta_{j_{n_m}} + \pi/2^{k_{j_{n_m}}}) \cup \limsup_m K((0,0), 2a_{j_{n_m}+1}\cdot t_{n_m})$. But the first upper limit is included in $p_{\Theta_0}$, since $k_n$ tends to infinity together with $n$ and the second is included in $\{(0,0)\}$, since $2a_{j_{n_m}+1}\cdot t_{n_m} \xrightarrow[m\to\infty]{} 0$, by virtue of $a_{j_{n_m}}\cdot t_{n_m} \le 1$ and $a_{j_{n_m}+1}/a_{j_{n_m}} \xrightarrow[m\to\infty]{} 0$.

To prove $2^0$ fix $\Theta \in [0,\pi)$. There exists a sequence $\{n_m\}_{m\in\mathbb{N}}$ tending to infinity such that $\Theta \in [\Theta_{n_m}, \Theta_{n_m} + \pi/2^{k_{n_m}})$ for each $m \in \mathbb{N}$. Thus $A \cap p_\Theta$ treated as a linear set is of the form $\bigcup_{m=1}^{\infty}[a_{n_m}, 2a_{n_m}]$, so $(0,0)$ obviously is neither $d_1$- nor $\mathcal{I}_1$-dispersion point of $A \cap p_\Theta$.□

Let us observe also that if we define the operators $\Phi : 2^{\mathbb{R}^2} \to 2^{\mathbb{R}^2}$ and $\Phi_{\mathcal{I}_2} : 2^{\mathbb{R}^2} \to 2^{\mathbb{R}^2}$ by the formulae:

$$\begin{aligned}\Phi(A) &= \{(x,y) \in \mathbb{R}^2 : (x,y) \in ap\text{-core } (A \cup \{x,y\})\}\\ \Phi_{\mathcal{I}_2}(A) &= \{(x,y) \in \mathbb{R}^2 : (x,y) \in \mathcal{I}ap\text{-core } (A \cup \{x,y\})\},\end{aligned}$$

then neither $\Phi$ nor $\Phi_{\mathcal{I}_2}$ is a lower density operator. Indeed, if $A = \mathbb{R}^2 \setminus E$, where $E$ is the Sierpiński set ([O], th. 14.4), then $\Phi(A) = \mathbb{R}^2$, so $A \triangle \Phi(A)$ is not of Lebesgue plane measure zero. Similarly, if $E$ is the set from th. 15.5 in [O], then for $A = \mathbb{R}^2 \setminus E$ we have $\Phi_{\mathcal{I}_2}(A) = \mathbb{R}^2$ and $A \triangle \Phi_{\mathcal{I}_2}(A)$ is not of the first category. Also if $A \triangle B$ is of Lebesgue plane measure zero (of the first category in the plane), then $\Phi(A)$ need not be equal to $\Phi(B)$ ($\Phi_{\mathcal{I}_2}(A)$ need not be equal to $\Phi_{\mathcal{I}_2}(B)$). To verify this it is sufficient to change $A$ essentially on one chosen straight line.

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