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LIMITS AND SUMS OF EXTENDABLE CONNECTIVITY FUNCTIONS

This paper shows how to express an arbitrary real function $f : [0,1] \to [0,1]$ as the pointwise limit of a sequence of extendable connectivity functions and as the sum of an infinite series of extendable connectivity functions. We use a result which describes the extendable connectivity functions $g : I \to I$ whose graphs are dense in I^2 in terms of certain subsets on which g may be redefined arbitrarily with values in I and still remain an extendable connectivity function. This result seems related to an open problem of whether or not the class of extendable connectivity functions can be characterized in terms of associated sets.

Let $I = [0,1]$. A function $G : X \to Y$ between topological spaces X and Y is *Darboux* if it maps connected sets to connected sets, and it is *almost continuous* if every open neighborhood of the graph of G in $X \times Y$ contains the graph of a continuous function from X into Y. We say G is a *connectivity* function if whenever C is a connected subset of X, then the graph of the restriction $G|C : C \to Y$ is a connected subset of $X \times Y$. A connectivity function $g : I \to R$ is *extendable* if there is a connectivity function $G : I \times I \to R$ for which $G(x,0) = g(x)$ when $0 \le x \le 1$.

Many relationships between different classes of functions defined here are already known. For functions $g : I \to I$, the connectivity functions are just the functions whose graphs are connected subsets of I^2. But for functions $G : I^2 \to I$, the connectivity functions are just the "peripherally continuous" functions [12], [19], [18]. Namely, for every $x \in I^2$ and every open neighborhood U of x and V of $G(x)$, there exists an open neighborhood W of x in U such that $G(\mathrm{bd}\, W) \subset V$. W and $\mathrm{bd}\, W$ can be chosen connected [18]. For functions $f : I \to R$, we have the following chain of classes of functions: continuous $\subset$ extendable connectivity $\subset$ almost continuous $\subset$ connectivity $\subset$ Darboux [18]. All of the containments are proper.

Every real function $f : R \to R$ is already known to be (1) the pointwise limit of a sequence of Darboux functions and (2) the sum of two Darboux

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functions [16], [7]. (1) and (2) are still true when "Darboux" is replaced by "connectivity" [17], [5] or by "almost continuous" [14]. In [8], Gibson asks if (1) and (2) are also true for "extendable connectivity" in lieu of "Darboux". He also asks if (3) the uniform limit of a sequence of extendable connectivity functions is again an extendable connectivity function. We give answers to his questions about (1) and (3), but give an alternate answer concerning his question about (2).

Here, we let K denote a class of functions from I into I, and we let $g \in K$. A set $M \subset I$ is called *g-negligible* with respect to K if $f \in K$ whenever $f : I \to I$ and $f = g$ on $I - M$. This is the same as saying that every function $f : I \to I$ obtained by arbitrarily redefining g on M is still a member of K.

In [1], Brown characterized in terms of g-negligible sets the members of K whose graphs are dense in I^2 when K is the class of connectivity functions. In [15], Kellum showed that Brown's characterization is still valid when K is instead the class of almost continuous functions. We show Brown's characterization also holds for when K is the class of extendable connectivity functions, but we obtain the stronger condition that the set A in statement (iii) of the theorem below can be chosen to be a G_δ instead of a countable set. Extendable connectivity functions $g : I \to I$, onto I, whose graphs are dense in I^2 have been constructed in [2] and [10]. We first need a lemma.

Lemma 1 *If $G : I^2 \to I$ is peripherally continuous, then for every $x \in I$ and all open neighborhoods U of $(x, 0)$ and V of $G(x)$, there exist an interval (a, b) about x in I and a connected open neighborhood W of $(x, 0)$ in U such that $(I \times \{0\}) \cap W = (a, b) \times \{0\}$, $(I \times \{0\}) \cap \operatorname{bd} W = \{a, b\} \times \{0\}$, and $G(\operatorname{bd} W) \subset V$.*

PROOF. There exists a connected open neighborhood O of $(x, 0)$ in U such that $\operatorname{bd} O$ is connected and $G(\operatorname{bd} O) \subset V$. There is an interval (a, b) about x such that $(I \times \{0\}) \cap O = (a, b) \times \{0\}$. Let $C_0 = (-\infty, a - 1] \cup [b + 1, \infty)$ and $C_n = [a - \frac{1}{n}, a - \frac{1}{n+1}] \cup [b + \frac{1}{n+1}, b + \frac{1}{n}]$ for $n = 1, 2, 3, \cdots$. Then $(C_n \times \{0\}) \cap \operatorname{bd} O$ is compact for each $n = 0, 1, 2, \cdots$. For each $y \in (C_n \times \{0\}) \cap \operatorname{bd} O$, there exists an open neighborhood W_y of y in U such that $\operatorname{diam} W_y < \frac{1}{n}$, $([a, b] \times \{0\}) \cap cl\, W_y = \emptyset$, and $G(\operatorname{bd} W_y) \subset V$. There exists a finite subcover $\mathcal{W}_n$ of the open cover $\{W_y : y \in (C_n \times \{0\}) \cap \operatorname{bd} O\}$ of $(C_n \times \{0\}) \cap \operatorname{bd} O$. Let W be the component of the open set $O - cl[\bigcup_{n=1}^{\infty} (\cup \mathcal{W}_n)]$ that contains $(x, 0)$. It follows that $\operatorname{bd} W \subset \operatorname{bd} O \cup (\cup\{\operatorname{bd} W^* : W^* \in \bigcup_{n=1}^{\infty} \mathcal{W}_n\})$, $(I \times \{0\}) \cap \operatorname{bd} W = \{a, b\} \times \{0\}$, and $G(\operatorname{bd} W) \subset G(\operatorname{bd} O) \cup (\cup\{G(\operatorname{bd} W^*) : W^* \in \bigcup_{n=1}^{\infty} \mathcal{W}_n\}) \subset V$.

Theorem 1 *Suppose K is the class of extendable connectivity functions and $g \in K$. Then the following are equivalent:*

(i) *The graph of g is dense in $I \times I$.*

(ii) *Every nowhere dense subset M of I is g-negligible.*

(iii) *There exists a dense G_δ subset A of I which is g-negligible.*

PROOF. (iii) $\Rightarrow$ (i): Suppose A is a dense G_δ subset of I which is g-negligible, and suppose $f : I \to I$ is any function for which $f = g$ on $I - A$. Then f is an extendable connectivity function, and so the graph of f is connected. This shows A is g-negligible with respect to the class of connectivity functions, which, according to Brown's result, implies g is dense in I^2.

(ii) $\Rightarrow$ (i): If every nowhere dense subset M of I is g-negligible with respect to the class of extendable connectivity functions, then M is g-negligible with respect to the class of connectivity functions. Therefore according to Brown's result, g is dense in I^2.

(i) $\Rightarrow$ (iii): The function $g : I \to I$ is extendable to a connectivity function $G : I^2 \to I$ and the graph of g is dense in I^2. For $n = 0, 1, 2, \ldots$, let $D_n = \{\frac{p}{2^n} :$ p is an integer and $0 \leq p \leq 2^n\} = \{d_1, d_2, \ldots, d_{2^n+1}\}$, and let $D = \bigcup_{n=0}^{\infty} D_n$, which is the set of dyadic rational numbers in $[0,1]$. Let $\{(x_n, g(x_n)) : n = 1, 2, 3, \ldots\}$ be a countable dense subset of g, and let $B = \{x_n : n = 1, 2, 3, \ldots\}$. Since G is peripherally continuous, then by the lemma, for each $x_n \in B$ and for each positive integer m, there exist an interval $I_{x_n,m}$ about x_n in I and a connected open neighborhood $U_{x_n,m}$ of $(x_n, 0)$ such that $\operatorname{diam} U_{x_n,m} < \frac{1}{m}$, $(I \times \{0\}) \cap U_{x_n,m} = I_{x_n,m} \times \{0\}$, $(I \times \{0\}) \cap \operatorname{bd} U_{x_n,m} = (\operatorname{bd} I_{x_n,m}) \times \{0\}$ and $\operatorname{diam}(G(\operatorname{bd} U_{x_n,m}) \cup G(x_n)) < \frac{1}{m}$. Let $E = \bigcup_{m,n=1}^{\infty} \operatorname{bd} I_{x_n,m}$.

Define $T_{n,m} = \{x \in I - E$: there exist 2^n+1 intervals $I_{y_1,m}, I_{y_2,m}, \ldots, I_{y_{2^n+1},m}$ such that $y_i \in B$, $\operatorname{diam} U_{y_i,m} < \frac{1}{m}$, $x \in \operatorname{int} I_{y_i,m}$, and $\operatorname{diam}(G(\operatorname{bd} U_{y_i,m}) \cup \{d_i\}) < \frac{1}{m}$ for $d_i \in D_n$ and $i = 1, 2, \ldots, 2^n + 1\}$. Each $T_{n,m}$ is a dense G_δ subset of I. By the Baire category theorem, $A = \bigcap_{n=0}^{\infty} \bigcap_{m=1}^{\infty} T_{n,m}$ is a dense G_δ subset of I.

Suppose $f : I \to I$ is a function for which $f = g$ on $I - A$. We show that the extension $F : I \times I \to I$ of f defined by

$$F(x,t) = \begin{cases} f(x) & \text{if } x \in A \text{ and } t = 0 \\ G(x,t) & \text{otherwise} \end{cases}$$

is peripherally continuous. Let $x \in I$, $\varepsilon > 0$, and let U be an open neighborhood of $(x, 0)$ in I^2 and $V = (F(x,0) - \varepsilon, F(x,0) + \varepsilon)$. Suppose $x \in A$. For some n, there exists $d_i \in D_n \cap V$. Choose a positive integer m so that $\frac{1}{m} < \varepsilon - |f(x) - d_i|$ and $W \subset U$ for each open neighborhood W of $(x, 0)$ in I^2 with diameter $< \frac{1}{m}$. Since $x \in T_{n,m}$, there exists $I_{y_i,m}$ such that $y_i \in B$, $\operatorname{diam} U_{y_i,m} < \frac{1}{m}$, $x \in \operatorname{int} I_{y_i,m}$, and $\operatorname{diam}(G(\operatorname{bd} U_{y_i,m}) \cup \{d_i\}) < \frac{1}{m}$. Therefore $U_{y_i,m} \subset U$ and $G(\operatorname{bd} U_{y_i,m}) \subset V$. $F((I \times \{0\}) \cap \operatorname{bd} U_{y_i,m}) = G((\operatorname{bd} I_{y_i,m}) \times \{0\})$ because $(I \times \{0\}) \cap \operatorname{bd} U_{y_i,m} = (\operatorname{bd} I_{y_i,m}) \times \{0\} \subset E \times \{0\} \subset (I - A) \times \{0\}$. Then

$F(\operatorname{bd} U_{y_i,m}) = G(\operatorname{bd} U_{y_i,m}) \subset V$. This shows F is peripherally continuous at each $(x,0) \in A \times \{0\}$.

Now suppose $x \in I - A$. Since G is peripherally continuous at $(x,0)$, there are an interval J about x in I and a connected open neighborhood O of $(x,0)$ in U such that $G(\operatorname{bd} O) \subset V$, $(I \times \{0\}) \cap O = J \times \{0\}$, $(I \times \{0\}) \cap \operatorname{bd} O = (\operatorname{bd} J) \times \{0\}$, and $F((\operatorname{bd} O) - (I \times \{0\})) = G((\operatorname{bd} O) - (I \times \{0\})) \subset V$. Let $a \in \operatorname{bd} J$. Then $a \notin A$. Otherwise, if $a \in A$, then for every i there exists an arbitrarily small $U_{y_i,m}$ containing a such that $G(bd(U_{y_i,m}) - (I \times \{0\}))$ is near d_i and such that $bd(U_{y_i,m}) \cap \operatorname{bd} O \neq \emptyset$. Then $G((\operatorname{bd} O) - (I \times \{0\}))$ is near d_i, a contradiction. Therefore $\operatorname{bd} J \subset I - A$ implies $F(\operatorname{bd} O) = G(\operatorname{bd} O) \subset V$. This shows F is peripherally continuous at each $(x,0) \in (I - A) \times \{0\}$. F is peripherally continuous at all points of I^2 off the closed set $I \times \{0\}$ since G is.

(i) $\Rightarrow$ (ii). Suppose the extendable connectivity function $g : I \to I$ has a graph which is dense in $I \times I$, and suppose M is a nowhere dense subset of I. $\bar{M}$ is a 0-dimensional compact set. Without loss of generality, we may suppose $\bar{M} \subset (0,1)$. Let $D = \{d_1, d_2, d_3, \ldots\}$ denote the set of dyadic rational numbers in I. For each positive integer m, there exists a disjoint finite cover $\mathcal{I}_m$ of $\bar{M}$ such that each member I_{mi} of $\mathcal{I}_m$ is a closed interval of length $< \frac{1}{m}$ such that $g(\operatorname{bd} I_{mi}) = d_m$ and each member $I_{m+1,j}$ of $\mathcal{I}_{m+1}$ is a subset of the interior of a member I_{mi} of $\mathcal{I}_m$. Then $\bar{M} = \bigcap_{m=1}^{\infty} (\cup \mathcal{I}_m)$.

Let A_{1i} be an isosceles triangle in I^2 with base $I_{1i} \times \{0\}$ and altitude < 1. Let J_{1k} be the closure of a component of $I - (\cup \mathcal{I}_1)$, and let B_{1k} be an isosceles triangle in I^2 with base $J_{1k} \times \{0\}$ and altitude < 1. Suppose $m \geq 1$ and $I_{m+1,j} \subset I_{mi}$. Then let $A_{m+1,j}$ denote an isosceles triangle in A_{mi} with base $I_{m+1,j} \times \{0\}$ and altitude $< \frac{1}{m+1}$, and if $J_{m+1,k}$ denotes the closure of a component of $I_{mi} - (\cup \mathcal{I}_{m+1})$, let $B_{m+1,k}$ denote an isosceles triangle in A_{mi} with base $J_{m+1,k} \times \{0\}$ and altitude $< \frac{1}{m+1}$. For example, the picture for stages $m = 1$ and $m = 2$ might look like Figure 1, page 187.

According to [11], a connectivity function $f : I \to I$ is extendable if and only if f is extendable to a connectivity function from I^2 into I that is continuous off $I \times \{0\}$. It follows that for every positive integer m and for each $J_{mk}, g|J_{mk}$ is extendable to a connectivity function $G : B_{mk} \to I$ such that G is continuous off $J_{mk} \times \{0\}$. Notice that for $m > 1$, $g(\operatorname{bd} J_{mk})$ is either $\{d_m\}$ or $\{d_m, d_{m-1}\}$. We define G to be d_m on the slanted sides of each triangle A_{mi}. At each stage m, G can be extended by the Tietze extension theorem to a function that is continuous on any remaining points of I^2 outside the union of the finite collection of triangles A_{mi} and B_{mk} (i, k varying). By construction, the resulting extension $G : I^2 \to I$ of g is peripherally continuous.

Now suppose $f : I \to I$ and $f = g$ on $I - M$. Then $f = g$ on $I - \bar{M}$. We

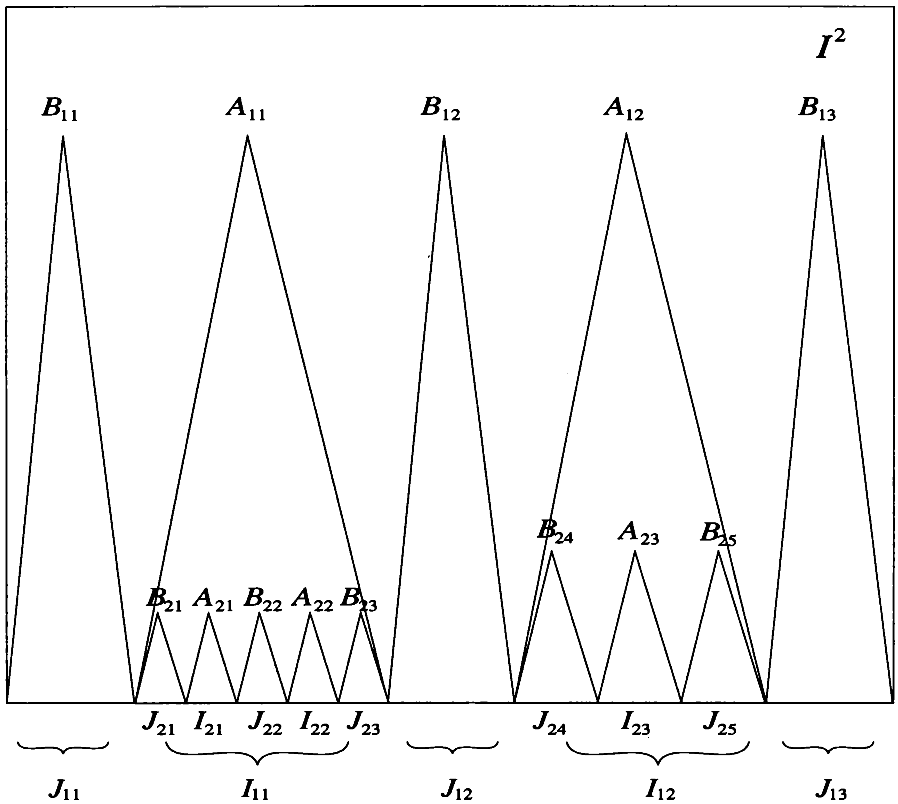

Figure 1

show that f is an extendable connectivity function by showing

$$F(x,t) = \begin{cases} G(x,t) & \text{on } I^2 - (M \times \{0\}) \\ f(x) & \text{on } M \times \{0\} \end{cases}$$

is a peripherally continuous extension of f. Since $F = G$ on the open set $I^2 - (\bar{M} \times \{0\})$, F is peripherally continuous at each point of $I^2 - (\bar{M} \times \{0\})$. So let $x \in \bar{M}$, $\varepsilon > 0$, and $\delta > 0$. Infinitely many dyadic rationals lie in $(F(x,0) - \varepsilon,\ F(x,0) + \varepsilon)$. Therefore there exist $d_m \in (F(x,0) - \varepsilon,\ F(x,0) + \varepsilon)$ and I_{mi} containing x such that the isosceles triangle A_{mi} with base $I_{mi} \times \{0\}$ has diameter $< \delta$ and $F(\mathrm{bd}\, A_{mi}) = G(\mathrm{bd}\, A_{mi}) = d_m$. This shows F is peripherally continuous at each point of $\bar{M} \times \{0\}$, too.

Remark 1 *It is not difficult to show that Brown's result is valid for the class*

K of Darboux functions. Theorem 1 implies that every function $f : I \to I$ is equal to each dense extendable connectivity function $g : I \to I$ except on some first category F_σ set $B = I - A$.

Not many characterizations of extendable connectivity functions are known. In [11], one is given in terms of a family of "peripheral intervals." For the class of Baire class 1 functions, the extendable connectivity functions are characterized in [3] as just the Darboux functions. A class K of real-valued functions f defined on an interval is said to be *characterized in terms of associated sets* if there exists a family P of subsets of R such that $f \in K$ if and only if for each $\alpha \in R$, the "associated" sets $E^\alpha(f) = \{x : f(x) < \alpha\}$ and $E_\alpha(f) = \{x : f(x) > \alpha\}$ belong to P. In [4], Bruckner showed the class K of Darboux functions cannot be characterized by associated sets. In [6], Christian and Tevy used Brown's result to show that the class K of connectivity functions $f : I \to I$ cannot be characterized in terms of associated sets. Then in [15], Kellum used his generalization of Brown's result to show that the class K of almost continuous functions $f : I \to I$ cannot be characterized by associated sets. Seeing a pattern here, we wonder if perhaps our generalization of Brown's result can be used to answer the following:

Question 1 *Can the class of extendable connectivity functions be characterized by associated sets?*

We now give an application of the first theorem.

Theorem 2 *Each function $f : [0,1] \to [0,1]$ is the pointwise limit of a sequence of extendable connectivity functions $f_n : [0,1] \to [0,1]$.*

PROOF. Let $g : [0,1] \to [0,1]$ be any extendable connectivity function onto $[0,1]$ whose graph is dense in $[0,1] \times [0,1]$. According to Theorem 1, there exists a dense G_δ subset A of $[0,1]$ on which we can redefine g arbitrarily with values in $[0,1]$ to obtain an extendable function. $[0,1] - A$ is an F_σ set and contains no interval. Therefore $[0,1] - A = \bigcup_{i=1}^\infty C_i$, where each C_i is nowhere dense in $[0,1]$ and if $i \neq j$, then $C_i \cap C_j = \emptyset$. For each $n \geq 1$, define $f_n : [0,1] \to [0,1]$ by

$$f_n = f|A \cup f|(\bigcup_{i=1}^{n} C_i) \cup g|(\bigcup_{i=n+1}^{\infty} C_i).$$

Then f_n is an extendable connectivity function because according to Theorem 1, we can redefine g arbitrarily on every nowhere dense subset M of $[0,1]$, such as $M = \bigcup_{i=1}^n C_i$ here, in order to obtain an extendable connectivity function again from $[0,1]$ into $[0,1]$. Let $x \in [0,1]$. If $x \in A$, $f_n(x) = f(x)$ for all n. But if $x \notin A$, there is an integer m such that $x \in C_m$, and $f_n(x) = f(x)$ for all $n \geq m$. This shows $f = \lim_{n\to\infty} f_n$.

It is a puzzlement whether f is the sum of just two extendable connectivity functions. We can, however, show the following.

Theorem 3 *Each function $f : [0,1] \to [0,1]$ equals a series $\sum_{n=1}^{\infty} g_n$ of extendable connectivity functions $g_n : [0,1] \to R$.*

PROOF. Let g, A, and C_n be as in the proof of Theorem 2. For each n, let $h_n : [0,1] \to [-2^{n-1}, 2^{n-1}]$ be a continuous onto function. Define

$$\begin{aligned}
g_1 &= \frac{1}{2}f|A \cup f|C_1 \cup g|(\bigcup_{i=2}^{\infty} C_i)\\
g_2 &= \frac{1}{4}f|A \cup 0|C_1 \cup (f-g)|C_2 \cup h_1 \circ g|(\bigcup_{i=3}^{\infty} C_i)\\
g_3 &= \frac{1}{8}f|A \cup 0|C_1 \cup 0|C_2 \cup (f - g - h_1 \circ g)|C_3 \cup h_2 \circ g|(\bigcup_{i=4}^{\infty} C_i)\\
g_4 &= \frac{1}{16}f|A \cup 0|C_1 \cup 0|C_2 \cup 0|C_3 \cup (f - g - h_1 \circ g - h_2 \circ g)|C_4\\
&\quad \cup h_3 \circ g|(\bigcup_{i=5}^{\infty} C_i)\\
&\vdots\\
g_n &= \frac{1}{2^n}f|A \cup 0|C_1 \cup 0|C_2 \cup \cdots \cup 0|C_{n-1}\\
&\quad \cup (f - g - h_1 \circ g - h_2 \circ g - \cdots - h_{n-2} \circ g)|C_n\\
&\quad \cup h_{n-1} \circ g|(\sum_{i=n+1}^{\infty} C_i) \text{ for } n = 3,4,5,\cdots.
\end{aligned}$$

Then $g_1 : [0,1] \to [0,1]$ and each $g_n : [0,1] \to [-2^{n-2}, 2^{n-2}]$, where $n > 1$, are extendable connectivity functions. For the case when $x \in A$, $\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(x) = f(x)$. But if $x \in C_m$, then $\sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{m} g_n(x) = f(x)$. Therefore $f = \sum_{n=1}^{\infty} g_n$.

The uniform limit of a sequence of almost continuous functions need not be Darboux [14]. It is not necessarily so, too, for the uniform limit of a sequence of extendable connectivity functions to be Darboux (or connectivity).

Example 1 *Let g, A, and C_n be as in the proof of Theorem 2. Define $f_n : [0,1] \to [0,1]$ as follows:*

$$f_1(x) = \begin{cases} g(x) & \text{if } x \notin A \cap g^{-1}\left(\frac{1}{2}\right) \\ 0 & \text{if } x \in A \cap g^{-1}\left(\frac{1}{2}\right) \end{cases}$$

$$f_2(x) = \begin{cases} f_1(x) & \text{if } x \notin C_1 \cap g^{-1}\left(\frac{1}{2}\right) \\ \frac{1}{2}+\frac{1}{2} & \text{if } x \in C_1 \cap g^{-1}\left(\frac{1}{2}\right) \end{cases}$$

$$f_3(x) = \begin{cases} f_2(x) & \text{if } x \notin C_2 \cap g^{-1}\left(\frac{1}{2}\right) \\ \frac{1}{2}+\frac{1}{3} & \text{if } x \in C_2 \cap g^{-1}\left(\frac{1}{2}\right) \end{cases}$$

$$\vdots$$

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{if } x \notin C_{n-1} \cap g^{-1}\left(\frac{1}{2}\right) \\ \frac{n+2}{2n} & \text{if } x \in C_{n-1} \cap g^{-1}\left(\frac{1}{2}\right) \end{cases} \quad \text{for } n = 2, 3, 4, \ldots.$$

This sequence of extendable connectivity functions f_n converges uniformly to a function $f : [0,1] \to [0,1]$ whose range is $[0,1] - \left\{\frac{1}{2}\right\}$.

[9] gives a Darboux function $f : [0,1] \to \mathbb{R}$ which is not the uniform limit of a sequence of connectivity functions, and [13] gives a connectivity function $f : [0,1] \to \mathbb{R}$ which is not the uniform limit of a sequence of almost continuous functions. Is there an example of an almost continuous function $f : [0,1] \to \mathbb{R}$ that is not the uniform limit of any sequence of extendable connectivity functions?

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