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ON THE TRANSFORMATIONS OF MEASURABLE SETS AND SETS WITH THE BAIRE PROPERTY

It is well known (see, e.g., [1], p. 901), that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence tending to zero and $A \subset [0,1]$ is a measurable set, then $\lim_{n\to\infty} \lambda(A\triangle(A+x_n)) = 0$, i.e. the sequence of characteristic functions of the sets $(A+x_n)$ converges in measure to a characteristic function of the set A. In [4] it was shown that for a set $A \subset [0,1]$ with the Baire property the situation is even better: the sequence of characteristic functions of the sets $(A+x_n)$ converges to a characteristic function of the set A except on a set of the first category. Also in [4] one can find an example showing that for measurable sets the convergence in measure cannot in general be improved to the convergence almost everywhere.

This paper is a continuation of [4]. We shall study the behaviour of the sequence of images $f_n(A)$ of a set $A \subset [0,1]$ when the sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous strictly increasing functions converges uniformly to the identity function (id).

Our first theorem together with its proof is completely analogous to theorem 2 in [4]. However, we shall present it below because: it is short, it is published in Russian and there is a possibility that this Georgian journal is not easily available now.

Theorem 1 *If a set $A \subset [0,1]$ has the Baire property and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous strictly increasing functions convergent uniformly to the identity function, then the sequence of characteristic functions of the sets $f_n(A)$ converges to a characteristic function of the set A except on a set of the first category.*

PROOF. In the sequel χ_E will denote the characteristic functions of a set E. The set A can be represented in the folowing way: $A = (G \cup Y) \setminus Z$, where G

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is an open set and Y, Z are sets of the first category. Put

$$P = Fr(G) \cup Y \cup Z \cup \bigcup_{n=1}^{\infty} f_n(Y) \cup \bigcup_{n=1}^{\infty} f_n(Z),$$

where $Fr(G)$ is a boundary of G. Observe that P is a set of the first category. We shall show that $\lim_{n\to\infty} \chi_{f_n(A)}(x) = \chi_A$ for each $x \in [0,1] \setminus P$. Indeed, let $x \in ([0,1]\setminus P)\cap A$. We have $\chi_A(x) = 1$. Obviously $x \in G$. Hence for sufficiently large n we have also $x \in f_n(G)$. For all n, $x \notin f_n(Z)$, so for almost all $n \in \mathbb{N}$, $x \in f_n(G)\cup f_n(Y)\setminus f_n(Z) = f_n(A)$ and finally $\lim_{n\to\infty} \chi_{f_n(A)}(x) = 1 = \chi_A(x)$. Suppose now that $x \in ([0,1] \setminus P) \setminus A$. We have $\chi_A(x) = 0$. Since $x \notin Fr(G)$, we have $x \notin f_n(G)$ for sufficiently large n. Also $x \notin f_n(Y)$ for all n, so $x \notin f_n(G) \cup f_n(Y)$ for almost all $n \in \mathbb{N}$. Then $x \notin f_n(G) \cup f_n(Y) \setminus f_n(Z) = f_n(A)$ for almost all $n \in \mathbb{N}$, whence $\lim_{n\to\infty} \chi_{f_n(A)}(x) = 0 = \chi_A(x)$.

Now we shall study the case of a measurable set $A \subset [0,1]$. Recall that each increasing function f can be represented as the sum of an absolutely continuous increasing function g and a singular increasing function h. This is known as a Lebesgue decomposition of a function f (cf. [2], theorem 8.13, p. 357). We shall always suppose additionally that $h(0) = 0$. The following lemma will be very useful in the proof of the main theorem:

Lemma 1 *Let $f : [0,1] \to \mathbb{R}$ be an increasing continuous function and $f = g + h$ its Lebesgue decomposition with g is absolutely continuous, h singular and $h(0) = 0$. If $B \subset \{x \in [0,1] : -\infty < f'(x) = g'(x) < +\infty\}$ is a measurable set such that $\lambda(B) = 1$ and $E = [0,1] \setminus B$, then*

$$\lambda(f(E)) = \lambda(h(E)) = h(1).$$

PROOF. It is well known that if some function F has a finite derivative on a measurable set D, then $F(D)$ is also measurable and $\lambda(F(D)) \leq \int_D |F'(x)|\, dx$ (see e.g., [2], theorem 8.7 p. 355 and theorem 8.10 p. 356). Hence $\lambda(h(B)) = 0$. Then

$$\begin{aligned}&\lambda(h([0,1])) = h(1) - h(0) = h(1) \leq \\ &\lambda(h(B)) + \lambda(h(E)) = \lambda(h(E)) \leq \lambda(h([0,1])),\end{aligned}$$

so $\lambda(h(E)) = h(1)$. To prove that $\lambda(f(E)) = \lambda(h(E))$ observe first that $f(B) \cap f(E) = \emptyset$ and $f([0,1]) = f(B) \cup f(E)$, so both sets are measurable. Further, from the absolute continuity of g and from the fact that $\lambda(B) = 1$ we conclude that

$$\begin{aligned} f(1)-f(0) &= \lambda\left(f([0,1])\right)=\lambda\left(f(B)\right)+\lambda\left(f(E)\right) \\ &\le \int_B f'(x)\,dx+\lambda\left(f(E)\right)=\int_B g'(x)\,dx+\lambda\left(f(E)\right) \\ &= \lambda\left(g(B)\right)+\lambda\left(f(E)\right)=g(1)-g(0)+\lambda\left(f(E)\right). \end{aligned}$$

Simultaneously, $f(1)-f(0)=g(1)-g(0)+h(1)$ so $\lambda(h(E))=h(1)\le\lambda(f(E))$.

Now let $G\supset f(B)$ be an arbitrary open set. Then $f^{-1}(G)\supset f^{-1}(f(B))=B$. Also

$$f^{-1}(G)=\bigcup_{n=1}^{\infty}(a_n,b_n),$$

where (a_n,b_n) are components of $f^{-1}(G)$. Hence

$$g(f^{-1}(G))=g\left(\bigcup_{n=1}^{\infty}(a_n,b_n)\right)=\bigcup_{n=1}^{\infty}g((a_n,b_n))\supset g(B)$$

It is easy to see that

$$\lambda(g(a_n,b_n))=g(b_n)-g(a_n)\le f(b_n)-f(a_n)=\lambda(f((a_n,b_n)))$$

for each $n\in\mathbb{N}$. So

$$\begin{aligned} \lambda\left(g(B)\right) &\le \lambda\left(g\left(\bigcup_{n=1}^{\infty}(a_n,b_n)\right)\right) \\ =\sum_{n=1}^{\infty}\left(g(b_n)-g(a_n)\right) &\le \sum_{n=1}^{\infty}\left(f(b_n)-f(a_n)\right) \\ =\lambda\left(\bigcup_{n=1}^{\infty}\left(f(a_n),f(b_n)\right)\right) &\le \lambda(G). \end{aligned}$$

From the arbitrariness of $G\supset f(B)$ we conclude that $\lambda(g(B))\le\lambda(f(B))$ and, we know that both sets $g(B)$ and $f(B)$ are measurable. Finally

$$f(1)-f(0)=g(1)-g(0)+h(1)=\lambda(g(B))+\lambda(h(E))=\lambda(f(B))+\lambda(f(E)),$$

so $\lambda(h(E))\ge\lambda(f(E))$, which ends the proof.

Corollary 1 $\lambda(f'(x)(B))=\int_B f'(x)\,dx=\int_B g'(x)\,dx$.

Theorem 2 *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous increasing functions convergent uniformly to the identity function. Then $\lim_{n\to\infty}\lambda^*(A\triangle f_n(A)) = 0$ for every measurable set $A\subset[0,1]$ if and only if for the sequences of terms $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ from the Lebesgue decomposition of $\{f_n\}_{n\in\mathbb{N}}$ the following conditions are fulfilled:*

1. *$\lim_{n\to\infty}h_n(1) = 0$ (i.e. $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly to 0)*
2. *the sequence $\{g_n\}_{n\in\mathbb{N}}$ consists of uniformly absolutely continuous functions (i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $n\in\mathbb{N}$ and for each finite collection of nonoverlapping intervals $[a_1,b_1],[a_2,b_2],\ldots,[a_k,b_k]$ contained in $[0,1]$ if $\sum_{i=1}^k(b_i-a_i)<\delta$, then $\sum_{i=1}^k(g_n(b_i)-g_n(a_i))<\varepsilon$).*

PROOF. Suppose first that (1) or (2) is not fulfilled. We shall show that there exists a measurable set $A\subset[0,1]$ for which $\lambda^*(A\triangle f_n(A))$ does not converge to 0.

Indeed, if (1) is not fulfilled, then there exists $\varepsilon > 0$ and an increasing sequence of natural numbers $\{n_m\}_{m\in\mathbb{N}}$ such that $h_{n_m}(1)\geq\varepsilon$ for each m.

Let $E_{n_m}\subset[0,1]$ be a set such that $\lambda(E_{n_m}) = 0$ and

$$\lambda(h_{n_m}(E_{n_m})) = h_{n_m}(1)$$

(it suffices to take $E_{n_m} = [0,1]\setminus\{x\in[0,1]: -\infty < f'_{n_m}(x) = g'_{n_m}(x) < \infty\}$). Put $A = \bigcup_{m=1}^{\infty}E_{n_m}$. We have obviously $\lambda(A) = 0$ and $f_{n_m}(A)\supset f_{n_m}(E_{n_m})$ for each m (both sets are measurable), so

$$\lambda(f_{n_m}(A))\geq\lambda(f_{n_m}(E_{n_m})) = \lambda(h_{n_m}(E_{n_m}))\geq\varepsilon,$$

whence $\lambda(A\triangle f_n(A))$ does not converge to 0.

Now suppose that (2) is not fulfilled. It means that there exist $\varepsilon_0 > 0$ such that for each positive integer m there exist n_m and a set I_m being finite union of nonoverlapping intervals such that $\lambda(I_m) < 1/m$ and $\lambda(g_{n_m}(I_m))\geq\varepsilon_0$. It is not difficult to observe that a sequence $\{n_m\}_{m\in\mathbb{N}}$ diverges to infinity. Choose an increasing sequence $\{m_p\}_{p\in\mathbb{N}}$ such that $\sum_{p=1}^{\infty}\frac{1}{m_p} < \frac{\varepsilon_0}{2}$. Put $A = \bigcup_{p=1}^{\infty}I_{m_p}$. We have $\lambda(A) < \frac{\varepsilon_0}{2}$ and

$$\lambda(f_{n_{m_p}}(A))\geq\lambda(f_{n_{m_p}}(I_{m_p}))\geq\lambda(g_{n_{m_p}}(I_{m_p}))\geq\varepsilon_0,$$

so $\lambda^*(A\triangle f_{n_{m_p}}(A))\geq\frac{\varepsilon_0}{2}$ for each $p\in\mathbb{N}$, whence $\lambda^*(A\triangle f_n(A))$ does not converge to 0.

Suppose now that (1) and (2) are fulfilled. Fix $\varepsilon > 0$. Let $\delta > 0$ be a number associated with $\varepsilon/4$ in (2). Let C be a finite union of disjoint intervals such that $\lambda(A\triangle C) < \min(\delta, \frac{\varepsilon}{4})$. For each positive integer n we have

$$\begin{aligned} A\triangle f_n(A) &\subset (A\triangle C) \cup (C\triangle f_n(C)) \cup (f_n(C)\triangle f_n(A)) \\ &= (A\triangle C) \cup (C\triangle f_n(C)) \cup (f_n(C\triangle A)). \end{aligned}$$

Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to the identity function we conclude that there exists $n_1 \in \mathbb{N}$ such that for $n > n_1$ we have $\lambda(C\triangle f_n(C)) < \frac{\varepsilon}{4}$.

Let $B_n = \{x \in [0,1] : -\infty < f_n'(x) = g_n'(x) < \infty\}$ and $E_n = [0,1] \setminus B_n$. We have

$$\begin{aligned} \lambda^*(f_n(C\triangle A)) &\le \lambda(f_n((C\triangle A)\cap B_n)) + \lambda^*((f_n(C\triangle A)\cap E_n)) \\ &\le \int_{(C\triangle A)\cap B_n} f_n'(x)\,dx + \lambda(f_n(E_n)) \\ &= \int_{(C\triangle A)\cap B_n} g_n'(x)\,dx + \lambda(f_n(E_n)) \end{aligned}$$

By Lemma 1 we conclude that $\lambda(f_n(E_n)) = \lambda(h_n(E_n))$ and since g is absolutely continuous, we have

$$\int_{(C\triangle A)\cap B_n} g_n'(x)\,dx = \lambda(g_n(C\triangle A)\cap B_n).$$

Since $\lambda((C\triangle A)\cap B_n) < \delta$, from the uniform absolute continuity we obtain easily $\lambda(g_n(C\triangle A)\cap B_n) < \frac{\varepsilon}{4}$. Let $n_2 \in \mathbb{N}$ be such that $h_n(1) < \frac{\varepsilon}{4}$ for $n > n_2$ Then for $n > n_2$ we have

$$\lambda(f_n(C\triangle A)) \le \lambda(g_n(C\triangle A)\cap B_n)) + \lambda(h_n(E_n)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Finally for $n > \max(n_1, n_2)$ we obtain $\lambda^*(A\triangle f_n(A)) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon$, whence $\lim_{n\to\infty} \lambda^*(A\triangle f_n(A)) = 0$.

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