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DIMENSION OF SETS OF NUMBERS WITH MULTIPLE REPRESENTATIONS

Given a natural number n, let $S(n) = \{0, 1, ..., n-1\}$. For each real number $b > 1$ let $S(n, b)$ be the set of all expressions of the form $\sum_{i=1}^{\infty} n_i \cdot b^{-i}$ with each n_i in $S(n)$ and with infinitely many of the n_i non-zero. The fact that each x in $(0, 1]$ has a unique representation, among those numbers not terminating in $0's$, in the base n number system can be stated as: 'each x in $(0, 1]$ has a unique representation in $S(n, n)$'.

We wish to consider writing $x \in (0, 1]$ with expressions in $S(n, b)$ where $b > 1$ is a real number with $n > b$. Let $U(n, b)$ be the set of x in $(0, 1]$ which have unique representations in $S(n, b)$; let $L(n, b)$ be the set of $x \in (0, 1]$ which can be represented in fewer than $\mathfrak{c}$ ways (where $\mathfrak{c}$ is the cardinality of the continuum). Unlike the situation in $S(n, n)$ where representations are unique, it turns out that for $n > b > 1$ the set $U(n, b)$ is of Lebesgue measure 0. The Hausdorff dimension of $U(n, b)$and $L(n, b)$ will be determined when b is an integer and estimates for the dimension of $U(n, b)$ will be obtained when b is not an integer. From the integral case one obtains a natural conjecture for $\dim(U(n, b))$ and $\dim(L(n, b))$.

We will need the following facts: If $A \subset S(n)$, then the Hausdorff dimension of $\{x : x = \sum n_i \cdot n^{-i}, n_i \in A\}$ has long been known to be $\log(k)/\log(n)$ where $k = \text{card}(A)$, the number of elements in A. (Cf.[1]; other results on expansions are obtained in [3].) Also, given a natural number N, $\{x : x = \sum n_i \cdot n^{-i},\ n_{i+N} \in A\}$ has dimension $\log(k)/\log(n)$ with $k = \text{card}(A)$. This follows from the similarity of these sets to the ones above.

Theorem 1 *Given natural numbers $b > 1$ and $n > b$, if $2b - 1 > n$, the dimension of $U(n, b)$ and that of $L(n, b)$ equal* $\log(2b-1)/\log(b)$*; if $n \geq 2b-1$, $L(n, b)$ is empty and hence $U(n, b)$ is also empty.*

Proof. First, let n and b be natural numbers with $1 < b \leq n < 2b - 1$. Consider the unique representation in $S(b, b)$ for a number $x \in (0, 1)$ as a

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sequence $\{n_i\}$ with each $n_i < b$. Suppose there is an i with $n_i < n - b$ and $n_{i-1} \neq 0$. Then the number x represented in $S(b,b)$ and in $S(n,b)$ as $n_0, ..., n_{i-1}, n_i, ...$ has in $S(n,b)$ a second representation; namely, $n_0, ..., n_{i-1} - 1, n_i + b, ...$ Clearly, each $x \in (0,1]$ does not belong to $L(n,b)$ (resp., $U(n,b)$) if its representation in $S(b,b)$ has infinitely many i (resp., at least one i) with $n_{i-1} = 0$ and $n_i < n-b$; for then x has $\mathfrak{c}$ (resp., at least two) representations in $S(n,b)$. To see that this is essentially the only way in which two representations can occur, fix a representation of $x = \sum n_i \cdot b^{-i}$ with $0 \leq n_i < n$ with infinitely many $n_i \neq 0$. Suppose $\sum n_i' \cdot b^{-i}$ is another representation for x. Let j be the least natural number with $n_j \neq n_j'$. Without loss of generality, suppose $n_j < n_j'$. Since each n_i', $i > j$ satisfies $n_i' \leq 2b - 2$, $\sum_{i>j} n_i' \cdot b^{-i} \leq (2b-2)\sum_{i>j} b^{-i} = 2b^{-j}$. Since $\sum_{i>j} n_i \cdot b^{-i} > 0$, $\sum_{i>j}(n_i' - n_i)b^{-i} < 2b^{-j}$. Thus n_j can differ from n_j' by at most 1 and $n_j = n_j' - 1$. Thus if x has two representations $x = n_o, ..., n_{i-1}, n_i, n_{i+1}, n_{i+2}, ...$ and $x = n_o, ..., n_{i-1} - 1, n_i + b, n_{i+1}, n_{i+2}, ...$ or $x = n_o, ..., n_{i-1} - 1, n_i + b - 1, n_{i+1} + b, n_{i+2}, ...$ or $x = n_o, ..., n_{i-1} - 1, n_i + b - 1, n_{i+1} + b - 1, n_{i+2} + b, n_{i+3}, ...$ or, in general, $x = n_o, ..., n_{i-1} - 1, n_i + b - 1, ..., n_{i+k} + b, n_{i+k+1}, ...$ There is also the possibility that x has a representation which terminates in $n - b - 1$'s, but the x's which have only this for a second representation form a countable subset of $(0,1]$. Thus except for the countable set of points which terminate in $n_i = n - b - 1$, a number x has more than one representation iff it has a non-zero n_{i-1} followed by some later $n_{i+k} < n - b$. Each n_{i-1} can be chosen to be the term preceding the term which is increased by b. Also except for the countable set which terminate in $n - b$'s, a number x has $\mathfrak{c}$ representations iff it has an infinite sequence of terms which are alternately non-zero and less than $n - b$. That is, $U(n,b)$ consists of those x in $S(b,b)$ which have no $n_i < n - b$ when $n_{i-1} \neq 0$. For each natural number N, let $A_N = \{x \in (0,1] : x = \sum_{i=1}^{\infty} n_{N+i} b^{-(N+i)}, n_{N+i} \neq 0, n_{N+i} \in S(b)\}$ and let $A_N' = \{x \in A_N :$ each $n_{N+i+1} \neq 0, 1, ..., n - b - 1\}$. Then $\dim(A_N') = \log(k)/\log(b)$ where $k = b - (n - b) = 2b - n$. Since $U(n,b)$ differs from $\cup_N A_N'$ by an at most countable set, $\dim(U(n,b)) = \log(2b - n)/\log(b)$. In order for x to belong to $L(n,b)$, when x is in A_N, there can be only finitely many n_i, $i > N$ for which $n_i = 0, 1, ... n - b - 1$. Thus if for each natural number M, $A_{N,M} = \{x \in A_N :$ for $i \geq M, n_i \neq 0, 1, ..., n - b - 1\}$, then $L(n,b)$ differs from $\cup A_{N,M}$ by an at most countable set and since each $A_{N.M}$ has dimension $\log(2b-n)/\log(b)$, this is also the dimension of $L(n,b)$. Now consider $n \geq 2b-1$. If $n_o, ..., n_{i-1}, n_i, ...$ is the representation for x in $S(b,b)$ and if $n_o = ... = n_N = 0$ and $n_{i-1} \neq 0$, then $n_o, ..., n_{i-1} - 1, n_i + b, ...$ is a second representation for x in $S(n,b)$ because $n_i < b$ implies $n_i + b < n$. By varying the representations of x through a sequence of such n_i, it is clear that each x in $(0,1]$ has $\mathfrak{c}$ representations in $S(n,b)$ when $n \geq 2b - 1$ and hence $U(n,b) = L(n,b) = \phi$.$\square$

A natural conjecture is that the statement of this theorem holds true for all n and b where $n > b > 1$ when b is a non-integral real number and n is a natural number. We will now obtain some estimates for the dimension of $U(n, b)$ consistent with this conjecture. These suggest more precise estimates and how to deal with $L(n, b)$. They involve geometrical means for dealing with these sets rather than the numerical ones in the theorem above.

Given n and b, let $c = b^{-1}$ and m be the greatest integer less than or equal to b. Let $w = \sum_{i=1}^{\infty}(n-1)b^{-i}$ and note that $w = (n-1)c/(1-c)$. It is worthwhile to draw a partition of the form $\{0, c, ..., kc, ..., mc, ...(n-1)c, nc\}$ for the interval $[0, nc]$. Clearly, each positive number can be written with an expression of the form $N = \sum n_i \cdot b^{-i}$ where N is a non-negative integer and each $n_i < n$. Moreover, each number in $[1, w]$ can be written either with $N = 0$ or with $N = 1$.

First consider $n > 2b - 1$. Then $w = (n-1)c/(1-c) > 2$. From this it follows that each $x \in (1, 2] \subset (1, w]$ can be written in two ways as $N + \sum n_i \cdot b^{-i}$. Likewise each $x \in (c, 2c]$ has two representations in $S(n, b)$, one with $n_1 = 0$, the other with $n_1 = 1$. Moreover, if $1 \leq k \leq n$, each $x \in (kc, (k+1)c]$ can be written with $n_1 = k - 1$ or $n_1 = k$. Similarly, each $x \in (kc^j, (k+1)c^j)$ can be written in two ways with $n_i = 0$ if $i < j$ and $n_j = k - 1$ or $n_j = k$. That is, each x in $(0, 1]$ can be written in more than one way when $n > 2b - 1$ and $U(n, b) = \phi$. (This argument shows that each x in $(0, 1]$ can be written in infinitely many ways and suggests that $L(n, b)$ is empty.)

Now consider $n < 2b - 1$. Since $w = (n-1)c/(1-c)$, one has $nc < w < 2$. Note that the sets $U(n, b) \cap (b^{-j}, b^{-j-1}]$ for $j = 1, 2, ...$ are similar geometrically and hence, in order to determine the dimension of $U(n, b)$ it suffices to consider $U(n, b) \cap (c, 1]$.

Since each $x \in (1, w]$ can be written in two ways as $N + \sum n_i \cdot b^{-i}$ (one with $N = 0$, the other with $N = 1$) each $x \in (c, cw]$ can be written in two ways (one with $n_1 = 0$, the other with $n_1 = 1$) and each $x \in (kc, (k-1)c + cs]$ can be written in two ways (one with $n_1 = k - 1$, the other with $n_1 = k$) $k = 1, ..., n - 1$.

In order to examine this case, it is worthwhile to draw a partition P of the form $\{0, c, ..., kc, ..., mc, ..., nc, (n+1)c, ..., 2(m+1)c\}$ which contains the interval $[0, 2]$. Since b is not an integer, $mc < 1 < (m+1)c$. Also, since $n < 2b - 1$ it follows that $(n+1)c > w$, and $nc < w$ so that $nc < w < (n+1)c$. Since each $x \in (1, w]$ can be written in two ways as $N + \sum n_i \cdot b^{-i}$ with $0 \leq n_i < n$, each x in each interval $(c, cw]$, $(2c, c + cw]$,...,$(mc, (m-1)c + cw]$ can be written in two ways with expressions from $S(n, b)$. That is, $U(n, b) \cap (0, 1]$ can be covered by m intervals of size $(2 - w)c$. (All of the last interval may not be needed.) Now, in each interval $(kc, (k+1)c]$ the interval $(kc, (k-1)c + cs]$ has been excluded from $U(n, b)$. Thus the remaining interval $((k-1)c + cs, (k+1)c]$

can be covered with $2(m+1)-n$ intervals (the number of intervals of P which meet the interval $(w,2]$). Each of these $2m-n+2$ intervals in the cover has length $(2-w)c^2$. (All of the first and last intervals may not be needed.) That is, $2m-n+2$ intervals each of length $(2-w)c^2$ cover $U(n,b) \cap (kc,(k+1)c]$. Thus $m(2m-n+2)$ intervals of size $(2-w)c^2$ cover $U(n,b)$. Since this process continues, $m(2m-n+2)^2$ intervals of size $(2-w)c^3$ cover $U(n,b)$ and in general $m(2m-n+2)^j$ intervals of size $(2-w)c^{j+1}$ cover $U(n,b)$. Let $s = \log(2m-n+2)/\log(b)$. Since $m \cdot (2m-n+2)^j((2-w)c^{j+1})^s = mc^s(2-w)^s(2m-n+2)c^{js} = mc^s(2-w)^s$, the s measure of $U(n,b)$ is less than or equal to $mc^s(2-w)^s$ and $\log(2m-n+2)/\log(b)$ is an upper bound for $\dim(U(n,b))$.

For a lower bound, note that if E_0 consists of the $m-1$ intervals of size $(2-w)c$ (excluding the last interval in the cover of $U(n,b)$ with m intervals) and E_1 consists of the $(m-1)(2m-n)$ intervals of size $(2-w)c^2$ (excluding the first and last intervals from the second cover of $U(n,b)$) and in general E_j consists of $(m-1)(2m-n)^j$ intervals of size $(2-w)c^{j+1}$ (excluded are the first and last intervals in each cover of $U(n,b)$ intersected with an interval of E_{j-1}) then $\cap E_j$ is contained in $U(n,b)$. A lower bound for $\dim(U(n,b))$ is $\log(2m-n)/\log(b)$. (Note that this makes sense only when $2m-n>1$.) That the dimension of the set $\cap E_j$ is $\log(2b-m)/\log(b)$ is somewhat intuitive. It does not seem to follow exactly from the arguments for standard Cantor sets (see e.g. [2] pages 14–17). The following proposition shows that this estimate holds.

Proposition 1 *Suppose $E = \cap E_J$ where for a non-negative number and natural number k*

i) *E_1 consists of a single interval,*

ii) *each E_{j+1} contains k intervals $I_1, \ldots, I_k$ of equal length in each interval I of E_j so that $k \cdot |I_k|^s = |I|^s$,*

iii) *there is $M < \infty$ so that if d_j is the minimum distance from an interval I of E_j to another interval I' of E_j, then $d_j > |I|/M$.*

Then $\dim(E) = s$.

PROOF. Clearly, for each natural number n, E is contained in k^n intervals of equal length l and by induction $k^n l^s = (\mathrm{diam}(E_1))^s$. Thus $\dim(E) \leq s$ and $s - m(E) \leq (\mathrm{diam}(E))^s$. To see that $\dim(E) \geq s$, let $s' < s$. Choose $\delta > 0$ so that when $d < \delta$, $d^{s'} > M^{s'} \cdot k \cdot d^s$. Let $\{J_i\}$ be a cover of E so that for each i, $|J_i| < \delta$. Let J be one of the J_i and let $I \subset E_j$ be the smallest interval in the construction of E which contains J. Let $I' \subset I$ be an interval of E_{j+1}. Then J contains points of two distinct intervals of E_{j+1} and hence $|J| > d_{j+1}$. Thus $|J|^{s'} > d_{j+1}^{s'} > (|I'|/M)^{s'} > k|I'|^s = |I|^s$. Therefore $\sum |J_i|^{s'} > \sum |I_i|^s \geq \text{diam}(E_1)^s$. Since for each $s' < s$ one has $s' - m(E) > 0$, it follows that $\dim(E) \geq s$. Hence $\dim(E) = s$.□

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