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Λ-VARIATION AND BAIRE CATEGORY

Abstract

Continuous functions of bounded λ-variation that are differentiable at least at one pointform a dense set of first Baire category in $C\Lambda BV$, the Banach space of continuous functions of bounded λ-variation . An example of a nowhere differentiable continuous function of bounded λ-variation is given. Furthermore, $C\Lambda BV$, as a subset of $C[0,1]$with the usual sup-norm, is a dense subset of first Baire category.

At the beginning of the 70s, D.Waterman extracted the useful concept of λ-variation from various techniques of the theory of Fourier series [8]. In this notewe will use the definitions and notations introduced in the fundamental paper[9].

Given a λ-sequence Λ, the set of all continuous functions of bounded Λ-variationis a closed linear subspace of the Banach space $(\Lambda BV, || \, ||_\Lambda)$, and will be denoted by $C\Lambda BV$. The set of functions differentiable at least at one point of $[a,b]$ will bedenoted by D. A λ-sequence $\Lambda = (\lambda_i)$ is said to be proper if $\lim \lambda_i = \infty$.

Proposition 1 *For any proper λ-sequence Λ, $D \cap C\Lambda BV$ is of first Baire category in $(C\Lambda BV, || \, ||_\Lambda)$.*

PROOF. We will follow the elegant idea of S.Banach [1]. Unfortunately, there is no suitable dense subset of $C\Lambda BV$ so that Banach's Satz 2 cannot be applied in our case. A slight modification of Banach's proof is required and careful construction of a "bad" function is necessary. Without loss of generality we may assume that the interval $[a,b]$ is $[0,1]$.

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For every positive integer m, we denote by Q_m^Λ the set of functions $x \in C\Lambda BV$ such that for some $t_0 \in [a,b]$ and for all $t \neq t_0$

$$\left|\frac{x(t) - x(t_0)}{t - t_0}\right| \leq m$$

Clearly, $D \cap C\Lambda BV \subset \bigcup_m Q_m^\Lambda$. We will show that each Q_m^Λ is nowhere dense which implies that $D \cap C\Lambda BV$ is a set of first Baire category as asubset of $C\Lambda BV$.

First we will show that Q_m^Λ is closed in $C\Lambda BV$. Suppose $x_n \in Q_m^\Lambda$, then $||x_n - x||_\Lambda \to 0$. By definition

$$\text{(1)} \qquad \forall n \quad \exists t_0^n \quad \forall t \neq t_0^n \quad \left|\frac{x_n(t) - x_n(t_0^n)}{t - t_0^n}\right| \leq m$$

Passing, if necessary, to a partial sequence, we can assume $t_0^n \to t_0$. Since convergence in $|| \; ||_\Lambda$-norm implies uniform convergence [9, p.42], we get $x_n(t_0^n) \to x(t_0)$. Passing to the limit in (1) for $n \to \infty$ yields

$$\left|\frac{x(t) - x(t_0)}{t - t_0}\right| \leq m$$

for all $t \neq t_0$, that is, $x \in Q_m^\Lambda$ as desired.

Now we shall show that Q_m^Λ is nowhere dense in $C\Lambda BV$. It suffices to show that given any $x \in Q_m^\Lambda$, no ball centered at x is contained in Q_m^Λ, that is,

$$\forall \epsilon > 0 \quad \exists \tilde{x} \notin Q_m^\Lambda \quad ||x - \tilde{x}||_\Lambda < \epsilon$$

Given such an x and an $\epsilon > 0$, take n such that $\frac{3m}{n}\sum_1^n \frac{1}{\lambda_i} < \epsilon$. We say that an interval $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ is of type A if

$$\exists t_0 \in I_i \quad \forall t \in I_i, t \neq t_0 \quad \left|\frac{x(t) - x(t_0)}{t - t_0}\right| \leq m$$

Now we will define an auxiliary function $y : [0,1] \to \mathbb{R}$. Set $y(0) = 0$. Suppose that y has been defined for I_i with$i \leq k$. To define y on I_{k+1} consider two cases.

If I_{k+1} is of type A, we set

$$y(\frac{k+1}{n}) = \begin{cases} 0 & \text{if } y(\frac{k}{n}) = \frac{3m}{n} \\ \frac{3m}{n} & \text{if } y(\frac{k}{n}) = 0 \end{cases}$$

and then define y to be continous and linear on I_{k+1}. Otherwise, we set $y(\frac{k+1}{n}) = y(\frac{k}{n})$ and define y to be linear and continuous on I_{k+1} (that is, constant). For every interval J with both endpoints of the form i/n either $|y(J)| = 0$ or $|y(J)| = 3m/n$.Since for every family $\{J_1, \ldots, J_j\}$ of nonoverlapping intervals with endpoints of the form i/n it must be $j \le n$, we get

$$\sum_1^j \frac{|y(I_{i_k})|}{\lambda_k} \le \sum_1^n \frac{\frac{3m}{n}}{\lambda_i} < \epsilon$$

Further, since all points of varying monotonicity of y are of the formi/n,we conclude that $||y||_\Lambda < \epsilon$ [5, Prop.1.1.]

Let $\tilde{x} = x + y$. Then $||x - \tilde{x}||_\Lambda < \epsilon$, and it remains to show $\tilde{x} \notin Q_m^\Lambda$, *i.e.*, we must show that

$$\forall t \quad \exists s \neq t \qquad \left|\frac{\tilde{x}(t) - \tilde{x}(s)}{t - s}\right| > m$$

Take any $t \in [0, 1]$. Then $t \in I_i$ for some i. If I_i is of type A and $t \neq t_0$,

$$\left|\frac{\tilde{x}(t) - \tilde{x}(t_0)}{t - t_0}\right| \ge \left|\left|\frac{y(t) - y(t_0)}{t - t_0}\right| - \left|\frac{x(t) - x(t_0)}{t - t_0}\right|\right| \ge 3m - m$$

If I_i is of type A and $t = t_0$, take any $s \in I_i$, $s \neq t_0$, and we get in a similar manner

$$\left|\frac{\tilde{x}(s) - \tilde{x}(t)}{s - t}\right| \ge 2m$$

If I_i is not of type A, then

$$\left|\frac{x(t) - x(s)}{t - s}\right| > m$$

for some $s \in I_i$ which completes the proof because $\tilde{x} = x$ on such an I_i. □

Remark What we have proven is in fact that $\bigcup_m Q_m^\Lambda$ is of first Baire category in $C\Lambda BV$. Observe that $\bigcup_m Q_m^\Lambda$ is the set of all $C\Lambda BV$-functions that have all Dini derivatives finite at at least one point. However, if necessary, one can slightly modify the above proof in order to show that a larger set of $C\Lambda BV$-functions that have both right-side Dini derivatives finite at at least one point is also of first Baire category (cf. Banach'sSatz 1).

Example *A nowhere differentiable continuous function that is of bounded λ-variation.*

To construct such example, it suffices to slightly alter the well-known van der Waerden function [7]. Of course, we have to assume that $\Lambda = (\lambda_i)$ is a proper

Λ-sequence. Let $u_0(x)$ be the distance from x to the nearest integer. Set $u_k(x) = 4^{-k}u_0(4^k x)$ for $k = 1, 2, \dots$. Clearly,

$$V_\Lambda(u_k, [0,1]) = \frac{1}{2\cdot 4^k}\sum_1^{2\cdot 4^k}\frac{1}{\lambda_i} \longrightarrow 0 \qquad \text{as } k \to \infty$$

Next, select a subsequence (u_{k_p}) such that $\sum_p ||u_{k_p}||_\Lambda < 1$. Then $f = \sum_p u_{k_p}$ is nowhere differentiable, as can be proven in the standard way [2, p.496]. Finally, $f \in C\Lambda BV$ by the virtue of [6, Prop.VI.5], since $(C\Lambda BV, ||\,||_\Lambda)$ is a complete space [9, pp.41-42].

S.Perlman has shown that $C[0,1] = \bigcup_\Lambda C\Lambda BV$ where the union is taken over all λ-sequences [3, Thm.9].We endow $C[0,1]$ with the usual sup-norm.

Proposition 2 *For any λ-sequence Λ, $C\Lambda BV$ is of first Baire category in $(C[0,1], ||\,||)$.*

PROOF. We start with the obvious equality

$$C\Lambda BV = \bigcup_{n=1}^{\infty} B_{C\Lambda BV}(0,n)$$

where $B_{C\Lambda BV}(0,n)$ denotes the closed ball in $(C\Lambda BV, ||\,||_\Lambda)$ of radius n, centered at 0 (the constant function 0). Thus, it suffices to show $B_{C\Lambda BV}(0,n)$ is nowhere dense in $(C[0,1], ||\,||)$. Since $B_{C\Lambda BV}(0,n)$ is a closed subset of $(C[0,1], ||\,||)$, the proof will be completed as soon as we show that $C[0,1] \setminus C\Lambda BV$ is dense in $(C[0,1], ||\,||)$. Thus, it suffices to show that 0 is a $||\,||$-limit of $C[0,1] \setminus C\Lambda BV$-functions. Indeed, given $\epsilon > 0$, it is rather elamentary to construct a function $\tilde{x} \in C[0,1]$ such that $||\tilde{x}|| < \epsilon$ and $V_\Lambda(\tilde{x}) = +\infty$. $\square$

We complete this note with the observation that both first Baire category sets discussed above are nevertheless relatively large.

Proposition 3 *For any λ-sequence Λ:*

1. *$D \cap C\Lambda BV$ is dense in $(C\Lambda BV, ||\,||_\Lambda)$*
2. *$C\Lambda BV$ is dense in $(C[0,1], ||\,||)$*

PROOF. (1) It is clear that, for $x \in C\Lambda BV$, $||x_n - x||_\Lambda \to 0$ where

$$x_n(t) = \begin{cases} x(t) & t \le 1 - \frac{1}{n} \\ x(1-\frac{1}{n}) & t \ge 1 - \frac{1}{n} \end{cases}$$

[9, Theorem 4]. Obviously, $x_n \in C\Lambda BV$ for all n.

(2) This follows immediately from the inclusion Polynomials $\subset C\Lambda BV$. $\square$

S. Perlman and D. Waterman gave a complete characterization of the inclusion $\Lambda BV \subseteq \Gamma BV$ for two distinct λ-sequences Λ and Γ [4, Theorem 3]. In the next proposition, we examine the proper case $\Lambda BV \subsetneq \Gamma BV$ from the point of view of Baire category.

Proposition 4 *If* $\Lambda BV \subsetneq \Gamma BV$, *then* $C\Lambda BV$ *is of first Baire category in* $(C\Gamma BV, \|\ \|_\Gamma)$.

PROOF. This can be proven in a manner fully analogous to the proof of Proposition 2. The only non-trivial ajustment is required in the construction of the function $\tilde{x}$.

It is elementary that $\Lambda BV \subsetneq \Gamma BV$ implies the existence of a sequence $a_n \searrow 0$ such that $\sum a_n/\gamma_n = \infty$ and $\sum a_n/\lambda_n < \infty$. For a number $\delta > 0$,set $a_n^\delta = \min\{\delta, a_n\}$, and then

$$x_\delta(\frac{1}{n}) = \begin{cases} \sum_{k=1}^{n}(-1)^{k+1}a_k^\delta - \sum_{k=1}^{\infty}(-1)^{k+1}a_k^\delta & \text{for } t = \frac{1}{n} \\ 0 & \text{for } t = 0 \end{cases}$$

We extend x_δ continuously onto the whole interval $[0,1]$ by requiring that x_δ be linear on each interval $[\frac{1}{n+1}, \frac{1}{n}]$. Then

$$V_\Gamma(x_\delta) = \sum_1^\infty \frac{a_n^\delta}{\gamma_n} = +\infty \quad \text{and} \quad V_\Lambda(x_\delta) = \sum_1^\infty \frac{a_n^\delta}{\lambda_n} \longrightarrow 0$$

as $\delta \to 0$. Hence, by picking a suitable δ, we can take $\tilde{x}$ to be x_δ. $\square$

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