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EXTREME PROBABILITY SUBMEASURES ON 3 POINTS

1 Introduction

Let $(X, \mathcal{P})$ be a finite set and the algebra of all its subsets. The collection of probability submeasures $\mathcal{S}$ on $(X, \mathcal{P})$ form a compact, convex set, and the question is to determine the extreme points. This problem is due to J. Roberts and is indirectly related to the Control Measure Problem of D. Maharam Stone (see [1] and [2]). If X consists of two points, then it is easy to see that there are three extreme submeasures — the two point masses and the submeasure which is identically 1 (except of course on the empty set).

In this note, we answer the question for three points and demonstrate a few lemmas which apply in the general case. In particular, for three points there are twelve extreme submeasures. The following table gives the twelve extreme submeasures and their values on the nonempty subsets of $X = \{1, 2, 3\}$. Each row represents one of the submeasures. Each column represents one of the nonempty subsets of $\{1, 2, 3\}$. The values are read in the obvious way (e.g., $\eta_{13}(23) = 0.5$).

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	123	23	13	12	3	2	1
δ_{123}:	1	1	1	1	1	1	1
δ_{23}:	1	1	1	1	1	1	0
δ_{13}:	1	1	1	1	1	0	1
δ_{12}:	1	1	1	1	0	1	1
δ_3:	1	1	1	0	1	0	0
δ_2:	1	1	0	1	0	1	0
δ_1:	1	0	1	1	0	0	1
η_θ:	1	1	1	1	0.5	0.5	0.5
η_{23}:	1	1	0.5	0.5	0.5	0.5	0.5
η_{13}:	1	0.5	1	0.5	0.5	0.5	0.5
η_{12}:	1	0.5	0.5	1	0.5	0.5	0.5
η_ρ:	1	0.5	0.5	0.5	0.5	0.5	0.5

Section 2 contains results showing that the above twelve are extreme points. Section 3 contains results showing that convex combinations of the above twelve give all submeasures, hence implying that they are all the extreme points.

We include in this section a general result and its proof which illustrates the geometric-polyhedral approach to the problem, which is different from the methods in the subsequent sections.

Definition 1 *A probability submeasure μ on $(X, \mathcal{P})$ is a function on $\mathcal{P}$ satisfying*

1. i. $\mu(\emptyset) = 0 \leq \mu(A) \leq 1 = \mu(X)$ *(positive, probability)*

2. ii. $\mu(B) - \mu(A) \geq 0$ *whenever* $A \subseteq B$ *(monotonicity)*

3. iii. $\mu(A) + \mu(B) - \mu(A \cup B) \geq 0$ *(subadditivity)*

From the definition, one has the following general result and proof (for which we would like to thank Arthur Stone).

Proposition 1 (A.H.Stone) *The extreme probability submeasures, on a finite set, take only rational values.*

PROOF. Consider the set S of probability submeasures μ on the set X of n points. Then S can be regarded as a subset of $\mathbb{R}^{2^n}$, with one coordinate corresponding to each subset of X. We cut down the dimension to $2^n - 2 = k$ by omitting coordinates for $\emptyset$ and X (of course $\mu(\emptyset) = 0$ and $\mu(X) = 1$). It is then easy to see that S is the convex polyhedral subset of $\mathbb{R}^k$ determined by the inequalities in the definition. Now the boundary of this set consists of points where one or more of these inequalities become(s) equalities — necessarily linear. Any single such equality produces a face of codimension 1; two independent ones give a face of codimension 2; and so on. The extreme points, i.e., the vertices of the polyhedron S are precisely the points determined by $k = 2^n - 2$ independent equalities (forming k independent linear equations). Of course the other inequalities must also be satisfied. The coefficients in the resulting k linear equations take only the values 0, 1 and -1. For example, $\mu(A) = 0$ has one coefficient 1; $\mu(B) - \mu(A) = 0$ has one 1 and one -1; $\mu(A) + \mu(B) - \mu(A \cup B) = 0$ has two 1's and one -1. In each case, the coefficients of the other coordinates (sets) are 0. The "right-hand sides" of the equations are 1 and (mostly) 0. What matters of course is that all the terms are integers. So the solutions to these simultaneous equations, by Cramer's Rule, are rational. □

This approach suggests a rather brute force method (which could be computer implemented) for obtaining the extreme submeasures. For example, for $n = 4$, $k = 2^4 - 2 = 14$, list the basic inequalities, select 14 independent ones to convert to equalities and solve the resulting simultaneous equations. More realistically, this gives a method (again, computer implementable) to test if a given submeasure is actually an extreme point. (Check how many inequalities are actually equalities.) The number of inequalities as a function of n can be calculated and grows quite fast. First, one reduces the number of inequalities derived from the definition by showing (1) for monotonicity it is only necessary to work with sets which differ by one point, and (2) for subadditivity it is only necessary to work with inequalities for disjoint sets. Under these conditions, the number of inequalities for n points is $\frac{3^n - 2^{n+1} + 1}{2} + n \cdot (2^{n-1} - 1)$.

Section 2

The symbols μ, ϕ, ψ will be used to denote submeasures (even if they are actually measures). As a general notation, we will assume that μ is a convex combination of ϕ and ψ, that is; $\mu = a \cdot \phi + b \cdot \psi$ where $a, b > 0$ and $a + b = 1$.

The following lemma is obvious.

Lemma 2 *If μ is a submeasure which only takes on the values 0 and 1, then μ is an extreme submeasure.*

Corollary 3 *The seven submeasures δ_j are all extreme.*

For reasons which will become apparent as we proceed, we subsume the above as

Lemma 4 *If $\mu(A) = 0$, then the same property is true of both ϕ and ψ, i.e. $\phi(A) = 0 = \psi(A)$. If $\mu(A) = 1$, then the same property is true of both ϕ and ψ.*

We also use this to motivate the following

Definition 2 *μ is extreme on the set A if whenever μ is a convex combination of ϕ and ψ then $\phi(A) = \mu(A) = \psi(A)$.*

Lemma 5 *Suppose $A \subset B$ and $\mu(A) = \mu(B)$. Then both ϕ and ψ satisfy the same relation $\phi(A) = \phi(B)$, $\psi(A) = \psi(B)$*

PROOF. Using the monotonicity of submeasures the result is elementary from

$$\mu(A) = a \cdot \phi(A) + b \cdot \psi(A) \leq a \cdot \phi(B) + b \cdot \psi(B) = \mu(B) = \mu(A)$$

Corollary 6 *This gives for each η_α other than η_θ that if it is a convex combination $\eta_\alpha = a \cdot \phi + b \cdot \psi$, then both ϕ and ψ have the same 2-valued form.*

It is then simple to complete the proof that they are extreme.

PROOF THAT η_{23} IS EXTREME.

Assume $\eta = \eta_{23} = a \cdot \phi + b \cdot \psi$. Then we have $\eta(2) = \eta(21) = \eta(1) = \eta(31) = \eta(3) = 1/2$ and as stated above ϕ must take a single common value x on these sets, and ψ must take a single common value y. If $x > 1/2$ then $y < 1/2$. But then $\psi(23) = 1 \leq \psi(2) + \psi(3) < 1/2 + 1/2$.

We can make this general.

Lemma 7 *Assume A, B, C are three different sets satisfying $\mu(A) = \mu(B) = \mu(C) = \mu(A \cup B) = \mu(A \cup C) = \frac{1}{2}\mu(B \cup C)$. If μ is extreme on the set $B \cup C$, then it is extreme on the three sets and the other two unions.*

Lemma 8 *Assume $\mu(A \cup B) = \mu(A) + \mu(B)$. Then the same is true for ϕ and ψ.*

PROOF.

$$\begin{aligned}
\mu(A) + \mu(B) &= \mu(A \cup B) \\
&= a \cdot \phi(A \cup B) + b \cdot \psi(A \cup B) \\
&\leq a \cdot (\phi(A) + \phi(B)) + b \cdot (\psi(A) + \psi(B)) \\
&= \mu(A) + \mu(B).
\end{aligned}$$

□

To see that η_θ is extreme requires a slightly different proof.

PROOF THAT η_θ IS EXTREME. Assume $\eta = \eta_\theta = a \cdot \phi + b \cdot \psi$. Then ϕ and ψ are 1 on all sets other than the singletons. We start with $1/2 = \eta(1) = a \cdot \phi(1) + b \cdot \psi(1)$. If $\phi(1) > 1/2$ then $\psi(1) < 1/2$. Since ψ is a submeasure, we have $1 = \psi(12) \leq \psi(1) + \psi(2)$. Hence $\psi(2) > 1/2$ and similarly $\psi(3) > 1/2$. But then using $1/2 = \eta(2) = a \cdot \phi(2) + b \cdot \psi(2)$ and $1/2 = \eta(3) = a \cdot \phi(3) + b \cdot \psi(3)$ we conclude that $\phi(2) < 1/2$ and $\phi(3) < 1/2$. This is a contradiction.□

We can make this into a general lemma.

Lemma 9 *Assume A, B, C are three different sets satisfying $\mu(A \cup B) = \mu(A \cup C) = \mu(B \cup C) = x$ and that the measures $\mu(A) = \mu(B) = \mu(C) = x/2$. If μ is extreme on the three pairwise unions, then μ is extreme on the sets A, B, C.*

This completes everything needed to see that the above twelve are extreme. However, we would like to present a little more in this section which may have use in the more general case.

Definition 3 *Given the submeasure μ on X we define the weak-extreme graph as follows. The vertices are the non-empty subsets of X. An edge exists between the two sets A and B if either of the two conditions are satisfied.*

(i) *$A \subset B$ and $\mu(A) = \mu(B)$ (or $B \subset A$ etc.)*

(ii) *$A \subset B$ and $\mu(B) = \mu(A) + \mu(B \backslash A)$ (or B the subset)*

Note that the graph is not a directed graph so we don't care if A is the subset or superset. The significance of (i) is that if $A \subset B$ is not connected by property (i) then the size of A could be increased slightly without losing monotonicity (or the size of B decreased). The significance of (ii) is that if $A \subset B$ is not connected by property (ii) then then the size of A could be decreased slightly without losing subadditivity (or the size of the B increased).

The above definition is why we did not omit the value of the submeasures on the full set X even though this was always 1. That is, μ is extreme on X and so we "connect" to X.

Lemma 10 *If μ is extreme on the sets A and B and either*

1. $\mu(A \cup B) = \max(\mu(A), \mu(B))$

or

2. $\mu(A \cup B) = \mu(A) + \mu(B)$,

then μ is extreme on $A \cup B$

Proposition 11 *If the w-e graph is disconnected, then μ is not an extreme submeasure*

PROOF. Let $\mathcal{A}$ be a connected subgraph not containing X. Then for $\epsilon > 0$ chosen small enough

$$\mu_{\mathcal{A},\pm\epsilon} = \begin{cases} \mu(B) & \text{if } B \notin \mathcal{A} \\ (1 \pm \epsilon)\mu(A) & \text{if } A \in \mathcal{A} \end{cases}$$

are both probability submeasures.

Section 3

In this section, it is shown that convex combinations of the above twelve can give any desired submeasure. A submeasure is defined by its values on the six non-trivial subsets of $X = \{1, 2, 3\}$. So we are essentially working in a six-dimensional space and hence expect we will need at most seven of the above to get any desired submeasure. Thus this section is just a bit of "linear algebra". The goal however, is to have an orderly way of presenting the proofs which should "'extend" to the general case.

Throughout this section μ is an arbitrary submeasure which we show is a convex combination of the named extreme submeasures. We assign coefficients to the twelve submeasures as follows, and try to fit the resulting combination to μ. Thus we think of the following columns as summing to μ.

	123	23	13	12	3	2	1
$a_{123} \cdot \delta_{123}$	a_{123}	a_{123}	a_{123}	a_{123}	a_{123}	a_{123}	a_{123}
$a_{23} \cdot \delta_{23}$	a_{23}	a_{23}	a_{23}	a_{23}	a_{23}	a_{23}	0
$a_{13} \cdot \delta_{13}$	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	0	a_{13}
$a_{12} \cdot \delta_{12}$	a_{12}	a_{12}	a_{12}	a_{12}	0	a_{12}	a_{12}
$a_3 \cdot \delta_3$	a_3	a_3	a_3	0	a_3	0	0
$a_2 \cdot \delta_2$	a_2	a_2	0	a_2	0	a_2	0
$a_1 \cdot \delta_1$	a_1	0	a_1	a_1	0	0	a_1
$b_\theta \cdot \eta_\theta$	b_θ	b_θ	b_θ	b_θ	$0.5b_\theta$	$0.5b_\theta$	$0.5b_\theta$
$b_{23} \cdot \eta_{23}$	b_{23}	b_{23}	$0.5b_{23}$	$0.5b_{23}$	$0.5b_{23}$	$0.5b_{23}$	$0.5b_{23}$
$b_{13} \cdot \eta_{13}$	b_{13}	$0.5b_{13}$	b_{13}	$0.5b_{13}$	$0.5b_{13}$	$0.5b_{13}$	$0.5b_{13}$
$b_{12} \cdot \eta_{12}$	b_{12}	$0.5b_{12}$	$0.5b_{12}$	b_{12}	$0.5b_{12}$	$0.5b_{12}$	$0.5b_{12}$
$b_\rho \cdot \eta_\rho$	b_ρ	$0.5b_\rho$	$0.5b_\rho$	$0.5b_\rho$	$0.5b_\rho$	$0.5b_\rho$	$0.5b_\rho$
μ	1	x_{23}	x_{13}	x_{12}	x_3	x_2	x_1

For a fixed submeasure μ, each column can be summed, and by taking linear combinations of the summed columns we obtain the following relations.

(i) $a_1 = 1 - x_{23} - 0.5b_\rho - 0.5b_{12} - 0.5b_{13}$

(ii) $a_2 = 1 - x_{13} - 0.5b_\rho - 0.5b_{12} - 0.5b_{23}$

(iii) $a_3 = 1 - x_{12} - 0.5b_\rho - 0.5b_{13} - 0.5b_{23}$

(iv) $a_{12} = x_{13} + x_{23} - x_3 - 1 + 0.5b_\rho - 0.5b_\theta + 0.5b_{12}$

(v) $a_{13} = x_{12} + x_{23} - x_2 - 1 + 0.5b_\rho - 0.5b_\theta + 0.5b_{13}$

(vi) $a_{23} = x_{12} + x_{13} - x_1 - 1 + 0.5b_\rho - 0.5b_\theta + 0.5b_{23}$

(vii) $a_{123} = 1 - \Big((x_{12} + x_{13} + x_{23}) - (x_1 + x_2 + x_3)\Big) - b_\rho$
$- \Big(+0.5b_{12} + 0.5b_{13} + 0.5b_{23}\Big) + 0.5b_\theta$

For example, equation (i) can be obtained by subtracting the second column (for the set 23) from the first column (for the full set 123).

The problem is to show that for different submeasures μ the coefficients $a's$, $b's$ can be chosen positive and to sum to one. First we define some auxiliary functions.

(viii) Define $s = (x_{12} + x_{13} + x_{23}) - (x_1 + x_2 + x_3)$.

(ix) Define $s_{12} = x_{13} + x_{23} - x_3 - 1$.

(x) Define $s_{13} = x_{12} + x_{23} - x_2 - 1$.

(xi) Define $s_{23} = x_{12} + x_{13} - x_1 - 1$.

Our orderly way of analyzing the cases is via the functions s_{ij} and s. In the general case, there would be many more functions. The function s can be viewed as the difference between the sum of the size of the one-point sets and the sum of the size of the two point sets. It is easy to see that μ on n-points is a measure if and only if the sum of the size of the one point sets is 1.

In the sequel, we assume (without loss of generality) that $s_{12} \leq s_{13} \leq s_{23}$.

Lemma 12 $0 \leq s \leq 3/2$.

PROOF. $x_{ij} \geq x_i$ $\forall\ i, j$ and so $s \geq 0$. To get the right-hand inequality, sum up the three inequalities $x_{ij} - x_i - x_j \leq 0$ plus the three inequalities $x_i \leq 1$ and divide by 2.

Lemma 13 $s_{ij} \leq 1$.

PROOF. For s_{12} this follows from $x_{13} + x_{23} \leq 2$ and $x_3 + 1 \geq 1$.

Lemma 14 $s \geq 1 \Rightarrow s_{ij} \geq 0$.

PROOF. $0 \leq x_1 + x_2 - x_{12} \leq (s-1) + x_1 + x_2 - x_{12} = (x_{12} + x_{13} + x_{23} - x_1 - x_2 - x_3 - 1) + x_1 + x_2 - x_{12} = x_{13} + x_{23} - x_3 - 1 = s_{12}$

CASES

Case I. $1 \leq s \leq 3/2$. In this case, give the coefficients the following values.

$b_{ij} = 0$ and $b_\rho = 0$, $b_\theta = 2(s-1)$, $a_{123} = 0$, $a_i = 1 - x_{jk}$, $i \neq j \neq k$ and $a_{ij} = s_{ij} - (s-1) = x_i + x_j - x_{ij}$.

It is straightforward to check that these are all positive and sum to one.

Case IIa. $s \leq 1$ and $0 \leq s_{12} \leq s_{13} \leq s_{23}$

$b_{ij} = 0$, $b_\theta = 0$, $b_\rho = 0$, $a_i = 1 - x_{jk}$, $a_{ij} = s_{ij}$ and $a_{123} = 1 - s$

Case IIb. $s \leq 1$ and $s_{12} \leq 0 \leq s_{13} \leq s_{23}$

$b_{13} = 0$, $b_{23} = 0$, $b_\rho = 0$, $b_\theta = 0$, $b_{12} = -2s_{12}$, $a_{12} = 0$, $a_{13} = s_{13}$ and $a_{23} = s_{23}$, $a_1 = 1 - x_{23} + s_{12} = x_{13} - x_3$, $a_2 = 1 - x_{13} + s_{12} = x_{23} - x_3$, $a_3 = 1 - x_{12}$ and $a_{123} = 1 - s + s_{12} = x_1 + x_2 - x_{12}$

Case IIc. $s \leq 1$ and $s_{12} \leq s_{13} \leq 0$. s_{23} can be either positive or negative since only the differences ($s_{23} - s_{13}$ and $s_{13} - s_{12}$) are used.

$b_\theta = b_{13} = b_{23} = 0$, $0.5b_\rho = -s_{13}$, $0.5b_{12} = s_{13} - s_{12} = x_{12} + x_3 - x_{13} - x_2$, $a_1 = x_{13} - x_3$, $a_2 = x_{23} - x_3$, $a_3 = x_{23} - x_2$, $a_{12} = 0$ and $a_{13} = 0$, $a_{23} = s_{23} - s_{13} = x_{13} + x_2 - x_1 - x_{23}$, and $a_{123} = 1 - (a_1 + a_2 + a_3 + a_{23} + b_{12} + b_\rho) = x_{23} + x_1 - 1 \geq 0$

This completes the proof that the above twelve submeasures constitute all the extreme submeasures for three points.

Note added in proof: It has come to the attention of the authors that Komei Fukuda has written a program which calculates the extreme submeasures for n points.

References

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