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ON THE MEASURABILITY OF EXTREME PARTIAL $\mathcal{I}$-APPROXIMATE DERIVATIVES

Abstract

In this paper we prove that the extreme partial derivatives of a function having the Baire property, have the Baire property too.

Let $\mathbb{R}(\mathbb{R}^2)$ denote the real line (the plane) and let $\mathbb{N}$ the family of all positive integers. All topological notations are given with respect to the natural topology. We introduce the following notation:

$\mathcal{I}(\mathcal{I}^2)$ - the σ-ideal of subsets of $\mathbb{R}(\mathbb{R}^2)$ of the first category,

$\mathcal{S}(\mathcal{S}^2)$ - the σ -field of subsets of $\mathbb{R}(\mathbb{R}^2)$ having the Baire property.

We start with the definition of $\mathcal{I}$-density point which was introduced in [3].

Definition 1 [3] *We shall say that 0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if and only if for each sequence of positive integers $\{n_m\}_{m\in\mathbb{N}}$, there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that*

$$\{x : \chi_{n_{m_p}\cdot A\cap[-1,1]}(x) \not\to 1\} \in \mathcal{I}.$$

A point x_0 is a point of $\mathcal{I}$-density of a set $A \in \mathcal{S}$ if and only if 0 is a point of $\mathcal{I}$-density of the set $A - x_0$. A point x_0 is a point $\mathcal{I}$-dispersion of a set $A \in \mathcal{S}$ if and only if x_0 is a point of $\mathcal{I}$-density of the set $\mathbb{R} \setminus A$.

Definition 2 [1] *Let $F : \mathbb{R} \to \mathbb{R}$ have the Baire property in a neighborhood of x_0. The upper $\mathcal{I}$-approximate limit of F at x_0 ($\mathcal{I}$-$\limsup_{x\to x_0} F(x)$) is the greatest lower bound of the set $\{y : \{x : F(x) > y\}$ has x_0 as an $\mathcal{I}$-dispersion point$\}$. The lower $\mathcal{I}$- approximate limit, the right-hand and left-hand upper and lower $\mathcal{I}$-approximate limits are defined similarly. If $\mathcal{I}$-$\limsup_{x\to x_0} F(x) = \mathcal{I}$-$\liminf_{x\to x_0} F(x)$, their common value is called the $\mathcal{I}$-approximate limit of F at x_0 and denoted by $\mathcal{I}$-$\lim_{x\to x_0} F(x)$.*

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Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Put

$$U_{(x_0,y_0)}(x) = \frac{F(x, y_0) - F(x_0, y_0)}{x - x_0} \text{ for } x \in \mathbb{R}, x \neq x_0.$$

Definition 3 [1] *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be any function defined in some neighborhood of $(x_0, y_0) \in \mathbb{R}^2$ and having the Baire property there in the direction of the x-axis. We define the upper $\mathcal{I}$-approximate partial right derivative of F at (x_0, y_0) $(D^+_{I-ap}F_x(x_0, y_0))$ in the x direction as a corresponding extreme limit of $U_{(x_0,y_0)}(x)$ as x tends to x_0 from the right. The other extreme $\mathcal{I}$-approximate partial derivatives in the x direction are define similarly. If all of these derivatives are equal and finite, we call their common value the $\mathcal{I}$-approximate partial derivative of F at (x_0, y_0) and denote it by $\mathcal{I} - F'_x(x_0, y_0)$. In a similar way we can define the extreme $\mathcal{I}$-approximate derivative in the direction of the y-axis.*

Lemma 4 [2] *Let G be an open set of the real line; then 0 is an $\mathcal{I}$-dispersion point of G if and only if, for every $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and a real number $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i \in \{1, ..n\}$, there exist two numbers $j, j' \in \{1, .., k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) \cdot h,\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) \cdot h\right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk}\right) \cdot h,\ -\left(\frac{i-1}{n} + \frac{j'-1}{nk}\right) \cdot h\right) = \emptyset.$$

Lemma 5 *Let $H \in \mathcal{S}$. If 0 is an $\mathcal{I}$-dispersion point of H, then, for every $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and a real number $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i \in \{1, ..n\}$, there exist two numbers $j, j' \in \{1, .., k\}$ such that*

$$H \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) \cdot h,\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) \cdot h\right) \in \mathcal{I}$$

and

$$H \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk}\right) \cdot h,\ -\left(\frac{i-1}{n} + \frac{j'-1}{nk}\right) \cdot h\right) \in \mathcal{I}.$$

PROOF. Let $H \in \mathcal{S}$. Then there exist an open set G and two sets of the first category P_1, P_2 such that $H = (G \setminus P_1) \cup P_2$. If 0 is an $\mathcal{I}$-dispersion point of the set H, then 0 is an $\mathcal{I}$-dispersion point of the set G. Therefore, by Lemma 4, for every $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and a real number $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i \in \{1, ..n\}$, there exist two numbers $j, j' \in \{1, .., k\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) \cdot h,\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) \cdot h\right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk} \right) \cdot h,\ -\left(\frac{i-1}{n} + \frac{j'-1}{nk} \right) \cdot h \right) = \emptyset.$$

We observe that, for each open interval (a,b), if $G \cap (a,b) = \emptyset$, then $H \cap (a,b) \subset P_2 \in \mathcal{I}$. Therefore, for every $n \in \mathbb{N}$, there exist $k \in \mathbb{N}$ and a real number $\delta > 0$ such that, for any $h \in (0,\delta)$ and $i \in \{1,..n\}$ there exist two numbers $j,\ j' \in \{1,..,k\}$ such that

$$H \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk} \right) \cdot h,\ \left(\frac{i-1}{n} + \frac{j}{nk} \right) \cdot h \right) \in \mathcal{I}$$

and

$$H \cap \left(-\left(\frac{i-1}{n} + \frac{j'}{nk} \right) \cdot h,\ -\left(\frac{i-1}{n} + \frac{j'-1}{nk} \right) \cdot h \right) \in \mathcal{I}.$$

Theorem 6 *If a function $F : \mathbb{R}^2 \to \mathbb{R}$ has the Baire property, then the extreme $\mathcal{I}$-approximate partial derivatives also have the Baire property.*

PROOF. We shall only show that the function $D^+_{I-ap}F_x$ has the Baire property. By our assumption, there exists a residual subset Q of $\mathbb{R}^2$ such that $F_{|Q}$ is a continuous function. It is sufficient to show that, for each $a \in \mathbb{R}$, a set

$$A = \{(x,y) \in Q :\ D^+_{\mathcal{I}-\mathrm{ap}}F_x(x,y) < a\} \in \mathcal{S}^2.$$

We assume that $a \in \mathbb{R}$ and $A \notin \mathcal{I}^2$. Let $\{t_m\}_{m\in\mathbb{N}}$ be a increasing sequence of real numbers such that $\lim_{m\to\infty} t_m = a$.

Let $(x_0,y_0) \in \mathbb{R}^2$. We put $Q_{y_0} = \{x : (x,y_0) \in Q\}$ and

$$G_m(x_0,y_0) = \{x > x_0 : U_{(x_0,y_0)}(x) > t_m \text{ and } x \in Q_{y_0}\}.$$

By the continuity of the function $F_{|Q}$ we have that, for each $(x_0,y_0) \in Q$, the function $U_{(x_0,y_0)|Q_{y_0}}(x)$ is a continuous one at each point $x \neq x_0$ and therefore the set $G_m(x_0,y_0)$ is an open set relative to Q_{y_0}. Additionally, by Kuratowski-Ulam theorem we have that

$$V = \{y : Q_y \text{ is not a residual subset of } \mathbb{R}\} \in \mathcal{I}.$$

Therefore a set $C = \{(x,y) \in Q : y \in V\} \subset \mathbb{R} \times V \in \mathcal{I}^2$ and, for each $(x,y) \in Q \setminus C$, $G_m(x,y) \in \mathcal{S}$. For each $m \in \mathbb{N}$, let

$$A_m = \{(x,y) \in A \setminus C : x \text{ is a } \mathcal{I}\text{-dispersion point of the set } G_m(x,y)\}.$$

Then $A \setminus C = \bigcup_{m\in\mathbb{N}} A_m$.

Let $m \in \mathbb{N}$. By Lemma 5 and since $G_m(x,y) \in \mathcal{S}$, for any $(x,y) \in A_m$ and $n \in \mathbb{N}$, there exist $p,k \in \mathbb{N}$ such that, for any $0 < \delta < \frac{1}{p}$ and $i \in \{1,..,n\}$, there exists $j \in \{1,..,k\}$ such that $G_m(x,y) \cap I_{ijnk\delta}(x,y) \in \mathcal{I}$ where

$$I_{ijnk\delta}(x,y) = \left(\left(\frac{i-1}{n} + \frac{j-1}{n \cdot k} \right) \cdot \delta + x, \left(\frac{i-1}{n} + \frac{j}{n \cdot k} \right) \cdot \delta + x \right).$$

For any $n,k,p \in \mathbb{N}$ let

$$D_{mnkp} = \bigcap_{\delta \in (0,\frac{1}{p})} \bigcap_{i \in \{1,..,n\}} \bigcup_{j \in \{1,..,k\}} \{(x,y) \in A_m : \ G_m(x,y) \cap I_{ijnk\delta}(x,y) \in \mathcal{I}\}.$$

Then, by the above, $A_m = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \bigcup_{p \in \mathbb{N}} D_{mnkp}$. Let $n,m,k,p \in \mathbb{N}$, $\delta \in W_p$, $i \in \{1,..,n\}$ and $j \in \{1,..,k\}$. We put

$$Z = \{(x,y) \in A_m : \ G_m(x,y) \cap I_{ijnk\delta}(x,y) \in \mathcal{I}\}.$$

We shall show that Z is a closed set relative to $Q \setminus C$. Let $(x_0,y_0) \in ((Q \setminus C) \setminus Z)$. Then $G_m(x_0,y_0) \cap I_{ijnk\delta}(x_0,y_0) \notin \mathcal{I}$. Therefore there exists $x_1 \in G_m(x_0,y_0) \cap I_{ijnk\delta}(x_0,y_0) \cap Q_{y_0}$. Since the point $x_1 \in G_m(x_0,y_0)$, we have that $F(x_1,y_0) - F(x_0,y_0) > t_m \cdot t^*$ where $t^* = x_1 - x_0$. Let $\epsilon > 0$ be such that $F(x_1,y_0) - F(x_0,y_0) - 2 \cdot \epsilon > t_m \cdot t^*$. By the continuity of the function $F_{|Q}$ at (x_1,y_0) and since $(x_1,y_0) \in Q$, we have that there exists $\eta_1 > 0$ such that if $(x,y) \in (K((x_1,y_0),\eta_1) \cap Q)$, then $\mid F(x,y) - F(x_1,y_0) \mid < \epsilon$. By the continuity of the function $F_{|Q}$ in (x_0,y_0), there exists $\eta_0 > 0$ such that if $(x,y) \in K((x_0,y_0),\eta_0) \cap Q$, then $\mid F(x,y) - F(x_0,y_0) \mid < \epsilon$. Let $\eta = \min\{\eta_1,\eta_0\}$ such that

$$x_1 \in \left(\frac{(i-1)k+j-1}{nk}\delta + x_0 + \eta, \frac{(i-1)k+j}{nk}\delta + x_0 + \eta \right)$$

and

$$x_1 \in \left(\frac{(i-1)k+j-1}{nk}\delta + x_0 - \eta, \frac{(i-1)k+j}{nk}\delta + x_0 - \eta \right)$$

We observe that if $(x,y) \in K((x_0,y_0),\eta_1) \cap (Q \setminus C)$, then there exists an open interval (a,b) such that

$$((a,b) \cap Q_y) \times \{y\} \subset K((x_1,y_0),\eta) \cap (I_{ijnk\delta}(x,y) \times \{y\}).$$

Let $x' \in (a,b) \cap Q_y$ such that $x' - x < t^*$. Then

$$F(x',y) - F(x,y) > t_m \cdot t^* > t_m \cdot (x' - x).$$

Therefore there exists $x' \in G_m(x,y) \cap I_{ijnk\delta}(x,y) \cap Q_y$. Since the set $G_m(x,y)$ is an open set in Q_y, we have that $G_m(x,y) \cap I_{ijnk\delta}(x,y) \notin \mathcal{I}$, so $(x,y) \notin Z$.

We showed that $(Q \setminus C) \setminus Z$ is an open set in $Q \setminus C$, so Z is a closed set in $Q \setminus C$. Hence D_{mnkp} is also a closed set in $Q \setminus C$. Since $C \in \mathcal{I}^2$ we have that $D_{mnkp} \in \mathcal{S}^2$. Therefore, for each $m \in \mathbb{N}$, $A_m \in \mathcal{S}^2$ and $A \in \mathcal{S}^2$. Thus the proof of the theorem is completed.

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