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Liao Kecheng, Department of Mathematics & Statistics, University of Auckland, Private Bag, Auckland, New Zealand
Chew Tuan-Seng, Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511

ON CONVERGENCE THEOREMS FOR AP INTEGRALS

Abstract

In this note we prove a controlled convergence theorem for the AP integral by using a new version of generalized absolutely continuous functions.

1 Introduction

The approximately continuous Perron integral was first defined by Burkill in 1931. (See [2].) A Riemann-type definition of this integral can be given. (See [1, 4, 6].) A descriptive definition can also be given by characterizing the primitive of the integral in terms of a kind of generalized absolutely continuous functions. (See [1, 6, 9].) Recently Lee has defined another version of generalized absolutely continuous functions, which gives a descriptive characterization of the Henstock integral (See [6, page 129; 7].) A similar descriptive characterization of the approximately continuous Perron integral can be given. (See [3].) Recently, Liao and Sarkhel have independently pointed out to Bullen by correspondence that the definition of generalized absolutely continuous functions given in [1, Definition 9, page 246] does not characterize the approximately continuous Perron integral. The definition used in the characterization is too strong. Hence the controlled convergence theorem given in [10] or [6, page 144] for the AP integral is no longer general enough. In this note, we shall prove a controlled convergence theorem for the AP integral by using a new version of generalized absolutely continuous functions. The main idea of the proof is similar to that of the corresponding convergence theorem for the Henstock integral. (See [6, Theorem 21.3].) However, the proofs of Lemmas 4, 6 and 7 are made simpler.

Key Words: AP integral, approximately continuous Perron integral, Henstock integral, generalized absolutely continuous function, controlled convergence theorem
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2 Preliminaries

Let $x \in [a,b]$ and let U_x be a measurable subset with $[a,b]$ with $x \in U_x$ and of density 1 at x. In fact, we have defined a set-valued function from $[a,b]$ into $\{U_x; x \in [a,b]\}$. Let $\Delta(\{U_x\})$ be the collection of all interval-point pairs $([u,v],x)$ with $u,v \in U_x$ and $u \leq x \leq v$. The collection $\Delta(\{U_x\})$ is called an approximate full cover (AFC,in short) of $[a,b]$. (See [1;6,page 137].)

Let $\Delta_1 = \Delta_1(\{U_x^1\})$, $\Delta_2 = \Delta_2(\{U_x^2\})$ be two AFC's of $[a,b]$. Then Δ_2 is said to be finer than Δ_1, denoted by $\Delta_2 \leq \Delta_1$, if for each $x \in [a,b], U_x^2 \subset U_x^1$.

Let $\Delta = \Delta(\{U_x\})$ be a given AFC of $[a,b]$. Then it is known that for $i = 1,2,\dots,n$ there exists $([x_{i-1},x_i],\xi_i) \in \Delta$, such that $a = x_0 < x_1 < \cdots < x_n = b$. (See [6, page 137; 4, page 57].) The collection $\{([x_{i-1},x_i],\xi_i), i = 1,2,\dots,n\}$ is called a Δ-partition of $[a,b]$.

We denote a Δ-partition by $\{([u,v],\xi)\}$ or $\{(I,x)\}$ in which $[u,v]$ or I represents a typical interval $[x_{i-1},x_i]$, and ξ its associated point ξ_i. $(D)\sum$ denotes the sum over the partition D.

Let $D_1 = \{(I,x)\}$, $D_2 = \{(J,y)\}$ be two Δ-partitions. Then D_2 is said to be finer then D_1, denoted by $D_2 \leq D_1$, if for each J in D_2, there exists I in D_1 such that $J \subset I$. Also, $D_1 \backslash D_2$ denotes the collection of (I,x) such that I is a component interval of $J\backslash E_2$ and $(J,x) \in D_1$ where E_2 is the union of all intervals in D_2. Roughly speaking, $D_1\backslash D_2$ is D_1 with some missing subintervals and the missing part are those intervals from D_2. Obviously, $(D_1\backslash D_2)\sum |I| = (D_1)\sum |I| - (D_2)\sum |I|$ and $(D_1\backslash D_2)\sum F(I) = (D_1)\sum F(I) - (D_2)\sum F(I)$ where $I = [u,v]$ and $F([u,v]) = F(v) - F(u)$.

Let Δ be an AFC of $[a,b]$. Let $D \subset \Delta$. Then D is said to be a partial Δ-partition of $[a,b]$ if $\{I; (I,x) \in D\}$ is a collection of nonoverlapping subintervals of $[a,b]$.

A real-valued function F defined on $[a,b]$ is said to be $AC_{ap}^{**}(X)$, where $X \subset [a,b]$, if for every $\varepsilon > 0$, there exist an AFC Δ of $[a,b]$, and $\eta > 0$ such that for any two partial Δ-partitions D_1, D_2 of $[a,b]$ with the associated points in X and $D_2 \leq D_1$ satisfying $(D_1 \setminus D_2)\sum |I| < \eta$, we have $|(D_1 \setminus D_2)\sum F(I)| < \varepsilon$,. Here D_2 may be void.

In the above definition if we only consider one partial Δ-partition D_1 and D_2 is void, then F is said to be $AC_{ap}^{*}(X)$. Clearly, $AC_{ap}^{*}(X)$ is weaker than $AC_{ap}^{**}(X)$. We remark that we use $AC_{ap}^{**}(X)$, instead of $AC_{ap}^{*}(X)$, in the controlled convergence theorem. The idea of $AC_{ap}^{**}(X)$ is crucial. It is used in Lemma 4(*) and Lemma 5(**).

A sequence $\{F_n\}$ is said to be uniformly $AC_{ap}^{**}(X)$ if F_n is $AC_{ap}^{**}(X)$ but uniformly in n, i.e., $\eta > 0$ and Δ in the above definition with F replaced by F_n is independent of n.

A real-valued function F defined on $[a,b]$ is said to be ACG_{ap}^{**} if $[a,b] =$

$\cup_{i=1}^{\infty} X_i$ so that F is $AC_{ap}^{**}(X_i)$ for each i. Also, ACG_{ap}^{*} can be similarly defined. A sequence $\{F_n\}$ is said to be uniformly ACG_{ap}^{**} if $\{F_n\}$ is uniformly $AC_{ap}^{**}(X_i)$ for each i.

A real-valued function f is said to be AP integrable to A on $[a,b]$ if for every $\varepsilon > 0$, there is an AFC Δ of $[a,b]$ such that for any Δ-partition $D = \{([u,v],\xi)\}$ of $[a,b]$ we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon.$$

Theorem 1 *A function f is AP integrable on $[a,b]$ if and only if there exists an ACG_{ap}^{*} function F such that the approximate derivative $F'_{ap}(x) = f(x)$ almost everywhere in $[a,b]$.*

For a proof see [3, page 160].

Theorem 2 *If f is AP integrable on $[a,b]$ with primitive F, then F is ACG_{ap}^{**}.*

PROOF. It is known that if f is AP integrable on $[a,b]$, then f is measurable on $[a,b]$. Let $X = \{x; |f(x)| \le N\}$. Let $f_X(x) = f(x)$ when $x \in X$ and 0 otherwise. Then f_X is measurable and bounded. Hence f_X is McShane integrable over $[a,b]$, see [6, page 108]. In other words for every $\varepsilon > 0$, there exists $\delta(x) > 0$ such that for every δ-fine McShane partition $D = \{(I,x)\}$, we have $(D)\sum |F_X(I) - f_X(x)|I|| < \varepsilon$, where F_X is the primitive of f_X. Recall that a partition $D = \{(I,x)\}$ is said to be a δ-fine McShane partition if $[a,b]$ is the union of intervals I, $I \subset (x-\delta(x),\ x+\delta(x))$ where x need not belong to I. On the other hand there exists an AFC Δ such that $(D)\sum |F(I) - f(x)|I|| < \varepsilon$ whenever $D = \{(I,x)\}$ is a Δ-partition of $[a,b]$.

Let $U_x^1 = U_x \cap (x - \delta(x), x + \delta(x))$, where U_x is given as in Δ. Let $\Delta^1 = \Delta^1\{U_x^1\}$ be the approximate full cover induced by $\{U_x^1\}$. Now take any two partial Δ^1-partitions D_1, D_2 with associated points in X and $D_2 \le D_1$ such that $(D_1 \backslash D_2)\sum |I| \le \eta$. Note that $[u,v]$ is taken from $D_1 \backslash D_2$ and we use the associated point in D_1. Therefore $D_1 \backslash D_2$ is a δ-fine McShane partition. Then we have

$$\begin{aligned}
\left|(D_1 \setminus D_2)\sum F(I)\right| \le\ & (D_1)\sum |F(I) - f(x)|I|| + \\
& (D_1)\sum |F_X(I) - f_X(x)|I|| + \\
& (D_2)\sum |F(I) - f(x)|I|| + \\
& (D_2)\sum |F_X(I) - f_X(x)|I|| + \\
& (D_1 \setminus D_2)\sum |F_X(I) - f_X(x)|I|| + \\
& (D_1 \setminus D_2)\sum |f_X(x)|I|| \\
<\ & \varepsilon + \varepsilon + \varepsilon + \varepsilon + \varepsilon + N\eta.
\end{aligned}$$

Thus F is $AC^{**}_{ap}(X)$. Hence F is ACG^{**}_{ap}. □

3 Controlled Convergence Theorem

For $n = 1, 2, \cdots$ let f_n be an AP integrable functions on $[a, b]$ with primitive F_n. In this section we shall consider the following three conditions :

(i) $\{f_n\}$ converges to f almost everywhere on $[a, b]$, as $n \to \infty$,

(ii) $\{F_n\}$ is uniformly ACG^{**}_{ap} on $[a, b]$ and

(iii) $\{F_n\}$ converges to F on $[a, b]$, as $n \to \infty$.

Theorem 3 *Let conditions (i) and (ii) hold. Then f is AP integrable on $[a, b]$, and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

Before we prove this theorem, we need the following lemmas.

Lemma 1 *Let the condition (ii) hold. Then for each x, there exists M_x such that $|F_n(x)| \le M_x$ for all n.*

PROOF. If $\{F_n\}$ is uniformly ACG^{*}_{ap} on $[a, b]$, then $\{F_n\}$ is equi-approximately continuous on $[a, b]$. For each $x \in [a, b]$ there exists a measurable set U_x with density 1 at x such that $|F_n(y) - F_n(x)| \le 1$ for all n and all $y \in U_x$. Let $\Delta = \Delta\{U_x\}$ be the approximate full cover induced by $\{U_x\}$. Let $w \in [a, b]$. Then there exists a Δ-partition of $[a, w]$. Note that $F_n(a) = 0$ for all n. Hence there exists M_w such that $|F_n(w)| \le M_w$ for all n. □

Lemma 2 *Let condition (ii) hold. Then $\{F_n\}$ is uniformly VBG on $[a, b]$, i.e. there exist a sequence $\{Y_i\}$ of sets with $[a, b] = \cup_i Y_i$ and a sequence $\{M_i\}$ of positive integers such that $(D) \sum |F_n(u, v)| \le M_i$ for all n and all partial partitions $D = \{[u, v]\}$ of $[a, b]$ with u, v in Y_i.*

PROOF. Let $\{F_n\}$ be uniformly $AC^{*}_{ap}(X)$. Then there exist an AFC Δ of $[a, b]$ and $\eta > 0$ such that for any partial Δ-partition $D = \{([u, v], \xi)\}$ of $[a, b]$ with $\xi \in X$ and $(D) \sum |v - u| < \eta$, we have $(D) \sum |F_n(u, v)| < 1$ for all n. (We remark that in the definition of $AC^{*}_{ap}(X)$, we use $|(D) \sum F(u, v)| < \varepsilon$. However D is a partial Δ-partition¿. Hence we have $(D) \sum |F(u, v)| < 2\varepsilon$.) Let U_x be given as in Δ. Choose $\delta(x)$ with $1 \ge \delta(x) > 0$ such that for $0 < t < \delta(x)$ we have $|U_x \cap [x, x+t]| > t/2$ and $|U_x \cap [x-t, x]| > t/2$. Here $|A|$ denotes the outer measure of A. Let $E_i = \{x; \delta(x) > \frac{1}{i}\} \cap X$ for those positive integers i with $\frac{1}{i} \le \eta$. Let $E_{ij} = E_i \cap \left[a + \frac{j-1}{i}(b-a), a + \frac{j}{i}(b-a)\right]$

for $j = 1, 2, \cdots, i$. Let $x, z \in E_{ij}$ with $x < z$. Then $z - x \le \frac{1}{i} < \delta(x)$. Hence $|U_x \cap [x,z]| > (z-x)/2$ and $|U_z \cap [x,z]| > (z-x)/2$. It follows that

$$\begin{aligned} |z-x| + |U_x \cap U_z \cap [x,z]| &\ge |U_x \cap [x,z]| + |U_z \cap [x,z]| \\ &> |z-x|. \end{aligned}$$

Hence there exists $y \in U_x \cap U_z \cap [x,z]$. Consequently,

$$(D)\sum |F_n(x,z)| \le (D)\left\{\sum |F_n(x,y)| + \sum |F_n(y,z)|\right\}$$

for all partial partitions $D = \{[x,z]\}$ with $x, z \in E_{ij}$ and all n. Note that X is the union of all E_{ij} and $\{F_n\}$ is uniformly ACG_{ap}^{**}. Hence $\{F_n\}$ is uniformly VBG. □

Lemma 3 *Let condition (ii) hold. Then there exists a subsequence $\{F_{ni}\}$ of $\{F_n\}$ such that $\{F_{ni}\}$ converges on $[a,b]$.*

PROOF. In view of Lemmas 1 and 2, if $\{F_n\}$ is uniformly $AC_{ap}^{*}(X)$, then apply Helley's theorem (See [5, page 16].) and we obtain a subsequence of $\{F_n\}$ which converges on X. Next, apply the diagonal process and the result follows. □

We remark that, instead of proving Theorem 3, it is sufficient to prove Theorem 4 In fact, if only conditions (i) and (ii) are satisfied, then by Lemma 3 and Theorem 4, for each subsequence $\{f_{ni}\}$ of $\{f_n\}$, there exists a subsubsequence $\{f_{nij}\}$ such that $\lim_{j\to\infty} \int_a^b f_{nij} = \int_a^b f$. Hence $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

Theorem 4 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ satisfying (i), (ii) and (iii). Then f is AP integrable on $[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

First, we shall prove the following lemmas. Let f be a AP integrable function on $[a,b]$ and $X \subseteq [a,b]$. Define $f_X(x) = f(x)$ when $x \in X$ and $f_X(x) = 0$, otherwise. Let F_X be its primitive if f_X is AP integrable on $[a,b]$.

Lemma 4 *Let f be AP integrable on $[a,b]$ with primitive F. If F is $AC_{ap}^{**}(X)$, where X is a closed subset of $[a,b]$, then f_X is Lebesgue integrable on $[a,b]$.*

PROOF. Since F is $AC_{ap}^{**}(X)$, we can define $H_\Delta = \sup_D (D)\sum F(I)$ and $H = \inf_\Delta H_\Delta$, where the supremum is over all partial Δ-partitions $D = \{(I,x)\}$ with $x \in X$ and the union of I in D containing X. Let $\varepsilon > 0$. Choose Δ such that $0 \le H_\Delta - H < \varepsilon$. Next, choose an open set $G \supset X$ such that the outer measure $|G \setminus X| < \eta$, where η comes from the definition of $AC_{ap}^{**}(X)$

with given $\varepsilon > 0$. We may choose Δ finer than the AFC Δ_0 given in the definition of $AC^{**}_{ap}(X)$ and such that $I \subset G$ when $(I,x) \in \Delta$ and $x \in X$. Then choose a fixed partial Δ-partition $D_0 = \{(I,x)\}$ with $x \in X$ such that $H_\Delta - \varepsilon < (D_0)\sum F(I) \leq H_\Delta$. Since f is AP integrable on $[a,b]$, there is an AFC Δ_1 such that for any Δ_1-partition $D = \{(I,x)\}$ of $[a,b]$, we have $(D)\sum |f(x)|I| - F(I)| < \varepsilon$. We may choose $\Delta_1 \leq \Delta_0$ such that every Δ_1-partition is finer than D_0 and $I \cap X = \emptyset$ when its associated point $x \notin X$. Let $D = \{(I,x)\}$ be any Δ_1-partition of $[a,b]$. Let $D_1 = \{(I,x) \in D;\ x \in X\}$. Note that the union of intervals I in D_1 covers X. Thus $(D_0)\sum |I| - (D_1)\sum |I| < \eta$. Then

$$\begin{aligned}\left|(D)\sum\{f_X(x)|I| - H\}\right| &\leq \left|(D_1)\sum\{f(x)|I| - F(I)\}\right| + \\ &\quad \left|(D_0)\sum F(I) - H_\Delta\right| + \\ &\quad \left|(D_0)\sum F(I) - (D_1)\sum F(I)\right| + |H_\Delta - H| \\ &< \varepsilon + \varepsilon + \varepsilon + \varepsilon.\end{aligned}$$

Hence f_X is AP integrable. Denote the primitive of f_X by F_X. In view of

$$\begin{aligned}\left|(D)\sum F_X(I)\right| &\leq \left|(D)\sum\{F_X(I) - f_X(x)|I|\}\right| + \\ &\quad \left|(D)\sum\{F(I) - f_X(x)|I|\}\right| + \left|(D)\sum F(I)\right|,\end{aligned}$$

F_X is $AC^{**}_{ap}(X)$. Note that we may choose an AFC Δ such that $I \cap X = \emptyset$ when $(I,x) \in \Delta$ with its associated point $x \notin X$. Thus $F_X(I) = 0$. Consequently, F_X is $AC^{**}_{ap}[a,b]$. It is known that there exists a Δ-partition of $[u,v]$ for any $[u,v] \subset [a,b]$. Therefore F_X is absolutely continuous on $[a,b]$. Hence f_X is Lebesgue integrable on $[a,b]$. □

Lemma 5 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_n\}$ and $\{f_{n,X}\}$ a sequence of AP integrable functions on $[a,b]$ with primitives $\{F_{n,X}\}$ where $f_{n,X}(x) = f_n(x)$ when $x \in X$ and zero otherwise. If $\{F_n\}$ is uniformly $AC^{**}_{ap}(X)$, where X is closed, then for every $\varepsilon > 0$, there exists an AFC Δ, independent of n, such that for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in X$, we have $|(D)\sum\{F_{n,X}(I) - F_n(I)\}| < \varepsilon$ for all n.*

PROOF. Since $\{F_n\}$ is $AC^{**}_{ap}(X)$ uniformly, for every $\varepsilon > 0$, there exist an AFC Δ and $\eta > 0$, both independent of n, such that the rest of the condition for $AC^{**}_{ap}(X)$ holds. For each n, there exists an AFC Δ_n with $\Delta_n \leq \Delta$ such that for any Δ_n-partition $D = \{(I,x)\}$ of $[a,b]$, we have $(D)\sum |F_n(I) - f_n(x)|I|| < \varepsilon$ and $(D)\sum |F_{n,X}(I) - f_{n,X}(x)|I|| < \varepsilon$.

We may assume that $I \cap X = \emptyset$ when its associated point $x \notin X$ and $I \subset G$ if $x \in X$, where $G \supset X$, $|G \backslash X| < \eta$ and G is open. Let D be any partial Δ-partition of $[a,b]$ with associated points in X. Construct Δ_n-partition of each I in D and denote the total partition by D_1. Split D_1 into D_2 and D_3 so that D_2 contains the intervals with associated points in X and D_3 otherwise. Note that $(D \setminus D_2)\sum |I| = (D_3)\sum |I| < \eta$. Then for all n

$$\begin{aligned}
\left|(D)\sum\{F_{n,X}(I)-F_n(I)\}\right| &= \left|(D_1)\sum\{F_{n,X}(I)-F_n(I)\}\right| \\
&\le \left|(D_2)\sum\{\ \}\right| + \left|(D_3)\sum\{\ \}\right| \\
&\le \left|(D_2)\sum\{F_{n,X}(I)-f_{n,X}(x)(I)\}\right| + \\
&\quad \left|(D_2)\sum\{f_n(x)|I|-F_n(I)\}\right| + \\
&\quad \left|(D_3)\sum F_{n,X}(I)\right| + \left|(D_3)\sum F_n(I)\right| \\
&< \varepsilon+\varepsilon+0+\left|(D\backslash D_2)\sum F_n(I)\right| \\
&< \varepsilon+\varepsilon+0+\varepsilon
\end{aligned}$$ □

Lemma 6 *If the conditions in Lemma 5 are satisfied, then $\{F_{n,X}\}$ is uniformly absolutely continuous on $[a,b]$.*

PROOF. By Lemma 5 $\{F_{n,X}\}$ is uniformly $AC_{ap}^{**}(X)$. Hence $\{F_{n,X}\}$ is uniformly AC_{ap}^{**} on $[a,b]$. Thus $\{F_{n,X}\}$ is uniformly absolutely continuous. □

Lemma 7 *If the conditions in Lemma 5 are satisfied, and, in addition, $\{f_n\}$ converges to f almost everywhere on $[a,b]$, then f_X is Lebesgue integrable on $[a,b]$ and for every $\varepsilon > 0$, there is a positive integer N such that for any partial partition $D = \{I\}$ of $[a,b]$, we have $|(D)\sum\{F_{n,X}(I) - F_X(I)\}| < \varepsilon$ whenever $n > N$.*

PROOF. The assertion follows from Lemma 6 and Vitali's convergence theorem for Lebesgue integrals. □

Lemma 8 *If the conditions in Lemma 7 are satisfied, and, in addition, F_n converges to F on $[a,b]$, then for every $\varepsilon > 0$ there exists an AFC Δ such that for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in X$, we have*

$$\left|(D)\sum\{F_X(I) - F(I)\}\right| < \varepsilon.$$

PROOF. The assertion follows from Lemmas 5 and 7. □

Lemma 9 *Let $\{f_n\}$ be a sequence of AP integrable functions on $[a,b]$ satisfying condition (ii). Then the following approximately Lusin condition holds: for every set $Z \subset [a,b]$ of measure zero and for every $\varepsilon > 0$, there exists an AFC Δ of $[a,b]$ such that for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Z$, we have $|(D)\sum F_n(I)| < \varepsilon$ for all n.*

PROOF. Let $\{F_n\}$ be uniformly $AC^{**}_{ap}(X_i)$, where $[a,b] = \cup_i X_i$. Let $Z \subset [a,b]$ with $|Z| = 0$. Let $S_i = Z \cap X_i$. Then $|S_i| = 0$. Let $\varepsilon > 0$ and G_i an open set such that $S_i \subset G_i$ and $|G_i| < \eta_i$ where η_i comes from the definition of $AC^{**}_{ap}(X_i)$ for the given $\varepsilon 2^{-i}$. Let Δ_i be the corresponding AFC. We may assume that $[u,v] \subset G_i$ when its associated point $x \in S_i$. Let U^i_x be the density 1 set at x in Δ_i. Let $U_x = U^i_x$ if $x \in X_i \backslash X_{i-1}$, $i = 1, 2, \cdots$, where $X_0 = \emptyset$. Let Δ be the approximate full cover induced by $\{U_x\}$. Then for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Z$, we have $|(D)\sum F_n(I)| \le \sum_i \varepsilon 2^{-i} = \varepsilon$. □

Now we shall prove Theorem 4.

PROOF OF THEOREM 4 Let $\{F_n\}$ be uniformly $AC^{**}_{ap}(X_i)$, where $[a,b] = \cup_{i=1}^{\infty} X_i$. We may assume that $X_n \subset X_{n+1}$ for all n. Let Y_i be a closed subset of X_i such that $|Z| = 0$ where $Z = [a,b] \backslash \cup_{i=1}^{\infty} Y_i$. Let $\varepsilon > 0$. First, by Lemma 7, there exists an integer $n = n(i,j)$ such that for any partial partition $D = \{I\}$ of $[a,b]$, we have $|(D)\sum \{F_{n,Y_i}(I) - F_{Y_i}(I)\}| < \varepsilon 2^{-i-j}$. We may assume for each i that $\{n(i,j)\}$ is a subsequence of $\{n(i-1,j)\}$. For each $x \in Y_i$, there exists $m(x) = n(j,j)$ for some $j > i$ such that $|f_{m(x)}(x) - f(x)| < \varepsilon$. We may assume that $f_n(x)$ converges to $f(x)$ pointwise for each $x \in [a,b]$ and $f_n(x) = f(x) = 0$ for all n and all $x \in Z$.

There exists an AFC Δ_n such that for any partial Δ_n-partition $D = \{(I,x)\}$ of $[a,b]$, we have $(D)\sum |F_n(I) - f_n(x)|I|| < \varepsilon 2^{-n}$. By Lemmas 5, 8 and 9, there exists an AFC Δ, independent of n, such that for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Y_i$, we have $|(D)\sum \{F_{n,Y_i}(I) - F_n(I)\}| < \varepsilon 2^{-i}$ for all n, and $|(D)\sum \{F_{Y_i}(I) - F(I)\}| < \varepsilon 2^{-i}$. Furthermore, for any partial Δ-partition $D = \{(I,x)\}$ of $[a,b]$ with $x \in Z$, we have $|(D)\sum F_n(I)| < \varepsilon$ for all n and $|(D)\sum F(I)| < \varepsilon$. We may assume that $U_x \subset U_x^{m(x)}$ where U_x and $U_x^{m(x)}$ are as in Δ and $\Delta_{m(x)}$ respectively. Hence if $(I,x) \in \Delta$, then $(I,x) \in \Delta_{m(x)}$.

Let $D = \{(I, x)\}$ be any Δ-partition of $[a, b]$. Let D_2 be the subset of D such that $x \in Z$ and $D_1 = D \setminus D_2$. Let $W_i = Y_i \setminus Y_{i-1}$ and $Y_0 = \emptyset$. Then

$$\begin{aligned}
\left|(D)\sum f(x)|I| - F(I)\right| \leq\ & \left|(D_1)\sum \{f(x)|I| - f_{m(x)}(x)|I|\}\right| + \\
& \left|(D_1)\sum \{F_{m(x)}(I) - f_{m(x)}(x)|I|\}\right| + \\
& \left|(D_1)\sum_i \sum_{x\in W_i} \{F_{m(x)}(I) - F_{m(x),Y_i}(I)\}\right| + \\
& \left|(D_1)\sum_i \sum_{x\in W_i} \{F_{m(x),Y_i}(I) - F_{Y_i}(I)\}\right| + \\
& \left|(D_1)\sum_i \sum_{x\in W_i} \{F_{Y_i}(I) - F(I)\}\right| + \\
& \left|(D_2)\sum F(I)\right| \\
<\ & \varepsilon(b-a) + \sum_{n=1}^{\infty} \varepsilon 2^{-n} + \sum_{i=1}^{\infty} \varepsilon 2^{-i} + \\
& \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon 2^{-i-j} + \sum_{i=1}^{\infty} \varepsilon 2^{-i} + \varepsilon \\
\leq\ & \varepsilon(b - a + 5).
\end{aligned}$$

Hence f is AP integrable on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. Note that in the above

$$(D_1)\sum_i \sum_{x\in W_i} \{F_{m(x),Y_i}(I) - F_{Y_i}(I)\} =$$

$$(D_1)\sum_i \sum_j \sum_{m(x)=n(j,j), x\in W_i} \{F_{m(x),Y_i}(I) - F_{Y_i}(I)\}.$$

If $m(x) = n(j, j)$ and $x \in W_i$, then $j > i$. Hence $n(j, j) = n(i, k(j))$ for some $k(j)$. Therefore, the above sum is less than $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon 2^{-i-k(j)}$. □

We notice that the method of proof above relies very little on the properties of density sets. It is therefore natural to surmise that the results should also work for a more general system. Indeed, this is being worked out, and if it warrants publication, will appear elsewhere.

We remark that the following integral is defined by R. Gordon [4].

A function f is said to be Gordon integrable to A on $[a, b]$ if there exists $\Delta = \Delta(\{U_x\})$ such that for every $\varepsilon > 0$, there is a positive function $\delta(x)$ on

$[a, b]$ such that for any Δ^1-partition $D = \{([u, v], \xi)\}$ we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon,$$

where $\Delta^1 = \Delta^1(\{U_x^1\})$ and $U_x^1 = U_x \cap (x - \delta(x), x + \delta(x))$.

It is obvious that if f is Gordon-integrable on $[a, b]$, then f is AP integrable and their integrals coincide. Surprisingly, the converse is also true, though the proof is rather involved, see [8].

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