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ON VARIOUS POROSITY NOTIONS IN THE LITERATURE

A nomenclature for some porosity notions currently appearing in the literature is introduced. Using this nomenclature we show that most of the σ-ideals generated by these porosity notions are distinct in $\mathbb{R}$.

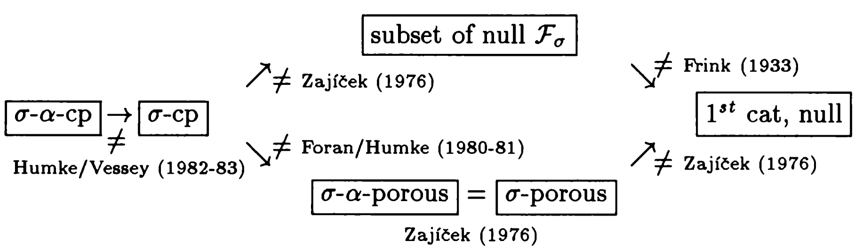
Let (X, d) be a metric space, $E \subseteq X$, $x \in X$, and $\delta > 0$. We define

$$\gamma(E,x,\delta) = \sup\{2\delta' : \ \delta' > 0 \text{ and } \exists\ x' \in X \text{ such that } B(x',\delta') \subseteq B(x,\delta) \text{ and } B(x',\delta') \cap [E \cup \{x\}] = \emptyset\}.$$

(If no such x' exists, we let $\gamma = 0$.) Letting

$$p(E,x) = \limsup_{\delta \to 0} \frac{\gamma(E,x,\delta)}{\delta},$$

we say that E is *porous* (*α-porous*) if $p(E,x) > 0$ ($p(E,x) \geq \alpha$) for all $x \in E$. In applications in convex geometry "for all $x \in E$" is frequently replaced by "for all $x \in X$". As this is equivalent to requiring that the closure of E be porous, we call this notion *closure porous* (cp). A σ-porous set is a countable union of porous sets, and likewise for other variants of porosity. The following diagram illustrates some of the relationships among these classes of sets for $X = \mathbb{R}$. (α is any fixed number satisfying $0 < \alpha < 1$.)



Remarks:

a. Frink's example [6] is in $\mathbb{R}^2$. In 1958 S. Marcus [9] gave examples in $\mathbb{R}^n$, $n \geq 1$.

b. The collection of subsets of $\mathcal{F}_\sigma$ null (i.e. Lebesgue measure zero) sets is equal to the σ-ideal generated by the Jordan measure zero sets.

Observe that E is α-porous if and only if given any $0 < \alpha' < \alpha$ and any $x \in E$, there is a sequence $\delta_n \searrow 0$ such that $\gamma(E, x, \delta_n) > \alpha'\delta_n$ for each n. This has the form "$\forall x \; \exists$ sequence." Interchanging the order of the quantifiers yields the logically stronger "$\exists$ sequence $\forall x$ " notion that we will call *globally porous.* (We mention that "globally porous" is sometimes used in the literature to denote what we are calling α-cp.) Specifically, we say that a set E is α-gp if and only if for each $0 < \alpha' < \alpha$ there is a sequence $\delta_n \searrow 0$ such that $\gamma(E, x, \delta_n) > \alpha'\delta_n$ for each n and for each $x \in E$.

Remarks (continued):

c. Global porosity is to porosity as uniform continuity is to continuity.

d. If E is α-gp, then E is α-cp.

e. There exists a closed symmetrically 1-porous set in $\mathbb{R}$ that cannot be expressed as a countable union of α-gp sets. (α is allowed to vary.)

f. An H set in the theory of trigonometric series is bilaterally α-gp for some $\alpha > 0$. (See the proof that H sets have no points of contraction on p. 384 of [3].)

g. Variations of the notion of global porosity appear in [15], [12], and [7]. In [12] Petukhov shows that for any $0 < \alpha' < \alpha \leq 1$, there is an α'-gp set in $\mathbb{R}$ that is not σ-α-gp. (The proof actually shows that the set is not σ-α-cp.)

h. Global versions of "very porous" (porosity using $\liminf$ rather than $\limsup$ in the definition) are used in [13], [4], [11], [2], and [10]. We note that such sets are both totally porous (in the sense of [1]) and superporous, and include (for $X = \mathbb{R}$) all symmetric Cantor sets having constant dissection ratios. (Hence not every gp set is an H set.)

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