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MULTIFRACTAL MEASURES

Let X be a metric space and μ a Borel probability measure on X. For $q, t \in \mathbb{R}$ and $E \subseteq X$ write

$$\overline{\mathcal{H}}_{\mu}^{q,t}(E) = \sup_{\delta>0} \inf\{\sum_i \mu(B(x_i,r_i))^q(2r_i)^t \mid (B(x_i,r_i))_i \text{ is a centered } \delta\text{-covering of } E\}$$

$$\overline{\mathcal{P}}_{\mu}^{q,t}(E) = \inf_{\delta>0} \sup\{\sum_i \mu(B(x_i,r_i))^q(2r_i)^t \mid (B(x_i,r_i))_i \text{ is a centered } \delta\text{-packing of } E\}$$

and put

$$\mathcal{H}_{\mu}^{q,t}(E) = \sup_{F\subseteq E} \overline{\mathcal{H}}_{\mu}^{q,t}(F)\,, \quad \mathcal{P}_{\mu}^{q,t}(E) = \inf_{E\subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu}^{q,t}(E_i)\,.$$

Then $\mathcal{H}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ are Borel measures - $\mathcal{H}_{\mu}^{q,t}$ is a multifractal generalization of the centered Hausdorff measure and $\mathcal{P}_{\mu}^{q,t}$ is a multifractal generalization of the packing measure. The measures $\mathcal{H}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ define, for a fixed q, in the usual way a generalized Hausdorff dimension $\mathrm{Dim}_{\mu}^{q}(E)$ and a generalized packing dimension $\mathrm{Dim}_{\mu}^{q}(E)$ of subsets E of X. We will discuss the functions

$$b_{\mu} : q \to \mathrm{Dim}_{\mu}^{q}(\mathrm{supp}\mu)\,, \quad B_{\mu} : q \to \mathrm{Dim}_{\mu}^{q}(\mathrm{supp}\mu)$$

and their relation to the so-called multifractal spectra functions of μ:

$$f_{\mu}(\alpha) = \mathrm{Dim}\{x \mid \lim_{r\searrow 0} \frac{\log \mu B(x,r)}{\log r} = \alpha\}\,,\ F_{\mu}(\alpha) = \mathrm{Dim}\{x \mid \lim_{r\searrow 0} \frac{\log \mu B(x,r)}{\log r} = \alpha\}$$

(here Dim and Dim denote the Hausdorff and packing dimension respectively).

In particular we discuss the relation between the dimension functions b_{μ} and B_{μ}, and the multifractal spectra functions f_{μ} and F_{μ} for random and non-random self-similar measures μ and self-affine measures μ in $\mathbb{R}^d$.