Real Analysis Exchange
Vol. 20(1), 1994/95, pp. 55–57

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HAUSDORFF AND PACKING MEASURES OF SOME SELF-AFFINE SETS

If $\{f_j\}_{j=1}^{\ell}$ are contracting affine maps in Euclidean space, then the unique compact set K satisfying $K = \bigcup_{j=1}^{\ell} f_j(K)$ is called a *self-affine* set. McMullen (1984) and Bedford (1984) determined the dimension of some specific self-affine sets in the plane. Fix integers $m < n$ and a "digit set" $D \subset \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, m-1\}$. The compact set

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k \middle| d_k \in D \right\}$$

is self-affine since

$$K(T, D) = \bigcup_{d \in D} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} (d + K(T, D)).$$

McMullen and Bedford showed that the Hausdorff dimension

$$\dim_H(K(T, D)) = \log_m \sum_{j=0}^{m-1} z(j)^{\alpha}$$

where $\alpha = \frac{\log m}{\log n}$ and

$$z(j) = \sum_{i=0}^{n-1} \mathbf{1}_D(i, j) = |D \cap \pi^{-1}(j)|.$$

On the other hand, the Minkowski (= "Box") dimension is

$$\dim_M[K(T, D)] = \log_m |\pi(D)| + \log_n \frac{|D|}{|\pi(D)|}$$

where π denotes projection to the second coordinate. An easy calculation shows that the Hausdorff and Minkowski dimensions of $K(T, D)$ coincide if and only if D has *uniform horizontal fibres*, i.e., all the positive values among $\{z(j)\}_{j=0}^{m-1}$ are equal. McMullen (1984) asked what is the Hausdorff measure of $K(T, D)$ in its dimension γ when D has nonuniform horizontal fibres. Gatzouras and Lalley (1992) showed that $H_\gamma[K(T, D)]$ must be either zero or infinity and Peres (1994a) later showed it is infinity. The zero-infinity dichotomy holds with respect to any gauge function and leads to the question of deciding for which measure functions φ is $H_\varphi[K(T, D)] = \infty$ and for which φ is $H_\varphi[K(T, D)] = 0$.

In Peres (1994a) we showed that in the nonuniform case the measure function

$$\varphi(t) = t^\gamma \exp\left[-c\frac{|\log\ t|}{(\log|\log\ t|)^2}\right]$$

yields ∞ (non σ-finite) Hausdorff measure for $K(T, D)$ if $c > 0$ is small enough, and

$$\varphi_\theta(t) = t^\gamma \exp\left[-\frac{|\log\ t|}{\log|\log\ t|^\theta}\right]$$

yields 0 Hausdorff measure provided $\theta < 2$. This still leaves a gap.

R.D. Mauldin (private communication) asked whether the dichotomy "zero or infinite Hausdorff measure" for $K(T, D)$ could be sharpened to "zero or non σ-finite". This would follow from a positive answer to the next question, which is still unsolved.

Question: Let K be an arbitrary self-affine set and φ any gauge function such that K is σ-finite for H_φ. Does it follow that in fact $H_\varphi(K) < \infty$?

The packing dimension of self-affine sets is always equal to the Minkowski dimension - this follows easily from results of Tricot (1982), see also Taylor and Tricot (1985). Peres (1994b) shows that the self-affine carpets $K(T, D)$ have infinite packing measure in their packing dimension when D has nonuniform horizontal fibres, and obtains a partial classification of gauge functions assigning $K(T, D)$ zero or infinite packing measure. The proof is based on the observation that most ϵ–disks in a canonical packing centered in $K(T, D)$ (with $\epsilon \asymp m^{-k}$) will have a very nonuniform distribution for the αk most significant digits in the base-m expansion of their y-coordinate, while the remaining $k - \alpha k$ digits will typically be approximately uniform. This enables combining packings of different sizes after an initial pruning of disks with nontypical expansion.

References

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