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LIMITING CASES OF THE SOBOLEV IMBEDDING THEOREM

The talk is based on the joint work with Professor D. E. Edmunds (University of Sussex, England) and Dr. B. Opic (Mathematical Institute in Prague, Czech Republic), papers [2], [3] and [4].

We provide estimates for an appropriate norm of the convolution of a function in a Lorentz space with one in a generalized Lorentz–Zygmund space. As a corollary it is shown that the Riesz potential of a function in an appropriate generalized Lorentz–Zygmund space satisfies a 'double exponential' integrability condition. The results extend those of Brézis–Wainger [1] on the convolution of functions in Lorentz spaces which lead to exponential integrability.

To be more precise we show one application, a refinement of the limiting case of the Sobolev imbedding theorem. It has been known that the Sobolev space $W^{1,n}(\Omega)$ (where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, $n \geq 2$) is (continuously) imbedded into the Lebesgue space $L^q(\Omega)$ (we write $W^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$) with $q \in [n, \infty)$ but is not imbedded into $L^\infty(\Omega)$. In 1967 N. S. Trudinger [6] proved the imbedding

$$W^{1,n}(\Omega) \hookrightarrow L_\Phi(\Omega) \tag{1}$$

where $L_\Phi(\Omega)$ is the Orlicz space with the Young function $\Phi(t) = e^{|t|^{n/(n-1)}} - 1$. It is known (see e.g. [5]) that the imbedding cannot be improved, i.e. taking $\Phi_\beta(t) = e^{\beta|t|^{n/(n-1)}} - 1$ with $\beta > n\omega_{n-1}^{1/(n-1)}$ (where $\omega_{n-1}^{1/(n-1)}$ denotes the surface of the unit sphere in $\mathbb{R}^n$) one can show that there exists a sequence $\{f_n\}$ of functions with norms $\|f_n\|_{W^{1,n}(\Omega)} \leq 1$ such that $\lim_{n\to\infty} \int_\Omega \Phi_\beta(|f_n|) = \infty$. This immediately implies that the imbedding (1) is not compact and that any imbedding $W^{1,n}(\Omega) \hookrightarrow L_\varphi(\Omega)$ with $\varphi(t)$ which increases strictly more rapidly than $\Phi(t)$ cannot hold. A natural question appears: *What is the substitution for the space $L_\Phi(\Omega)$ in the imbedding of the type (1) when we replace the space $W^{1,n}(\Omega)$ by some smaller one?* (Recall that the assumption $f \in W^{1,n+\varepsilon}(\Omega)$, $\varepsilon > 0$, implies that f is bounded on Ω and a.e. equivalent with a function continuous on Ω, i.e. $W^{1,n+\varepsilon}(\Omega) \hookrightarrow C_B(\Omega)$. So the space cannot be much smaller.) Our results enable to derive (for the sake of simplicity we restrict ourselves to a special case) the imbedding

$$W^{1,\Psi}(\Omega) \hookrightarrow L_\varphi(\Omega)$$

($W^{1,\Psi}(\Omega)$ has the norm $\|f\|_{W^{1,\Psi}(\Omega)} = \left(\|f\|_{L_\Psi(\Omega)} + \sum_{i=1}^{n} \left\|\frac{\partial f}{\partial x_i}\right\|_{L_\Psi(\Omega)}\right)$) with the Young functions Ψ, φ having the behaviors

$$\Psi(t) \approx |t|^n(1+\log|t|)^{n-1}, \quad t \to \infty,$$

$$\varphi(t) \approx \exp\exp(\beta|t|^{n/(n-1)}), \quad t \to \infty, \quad \beta > 0 \quad \text{sufficiently small.}$$

Similar sharpness results as for (1) are obtained for this case, too.

References

[1] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, Commun. Partial Differ. Equations **5** (1980), 773–789.

[2] D. E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalized Lorentz–Zygmund spaces*, Indiana Univ. Math. J. (to appear.)

[3] D. E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability, Bessel potentials and imbedding theorems* (in preparation.)

[4] D. E. Edmunds, P. Gurka and B. Opic, *Logaritmic Bessel potential spaces and sharpness of imbeddings* (in preparation.)

[5] J. A. Hempel, G. R. Morris and N. S. Trudinger,*On the sharpness of a limiting case of the Sobolev imbedding theorem*, Bull. Aust. Math. Soc. **3** (1970), 369–373.

[6] N. S. Trudinger*On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.