Real Analysis Exchange
Vol. 20(1), 1994/95, pp. 45–46

R. S. Ismagilov, Bauman State Technical University, Moscow, Russia

RANDOM WALKS AND GENERALIZED RIESZ PRODUCTS

The main subjects of this paper are tail σ-algebras of a random walk and the spectra of dynamical systems related to them.

1. Tail σ-algebras. Let $(\Omega, \mathcal{B}, P)$ be a probabilistic space, M a metric space and $f_n : \Omega \to M$, $n = 0, 1, 2, \dots$, measurable mappings. For any positive integer n denote by $\mathcal{B}_n$ a σ-algebra of $\mathcal{B}$ generated, first, by all sets $f_k^{-1}(D)$ (D is open in M and $k > n$) and, second, by all sets having a P - measure equal to 0. A σ-algebra $\mathcal{B}_{\text{tail}}$ (corresponding to the sequence $\{f_n\}$) is defined by $\mathcal{B}_{\text{tail}} = \bigcap_k \mathcal{B}_k$.

Example. If $\{f_n\}$ are independent random elements of M, then $\mathcal{B}_{\text{tail}}$ is trivial (i.e., it consists of only measurable sets $E \subset \Omega$ with $P(E)$ equal to either 0 or 1). (A. Kolomogorov)

2. The tail σ-algebra of a random walk. Now take $M = \mathbb{R}^q$ and consider independent random elements $\phi_k : \Omega \to \mathbb{R}^q$, $k = 0, 1, \dots$. The corresponding distributions are denoted by μ_k, $k = 0, 1, \dots$. The sequence $f_k = \phi_0 + \dots + \phi_k$, $k = 0, 1, \dots$, is, by definition, a random walk on $\mathbb{R}^q$. We are interested in the corresponding tail σ-algebra $\mathcal{B}_{\text{tail}}$. To pose the problem more definitely we define now the notion of a tail dynamical system.

Suppose that the measure μ_0 is equivalent to the Lebesgue measure ℓ on $\mathbb{R}^q$ and μ_k with $k > 0$ is supported by a Borel subset $Y_k \subset \mathbb{R}^q$. So we can put $\Omega = \mathbb{R}^q \times Y_1 \times Y_2 \times \dots$ and $f_k(\omega) = y_0 + \dots + y_k$, $k = 0, 1, \dots$, where $\omega = (y_0, y_1, \dots) \in \Omega$. For any $t \in \mathbb{R}^q$ define the transformation $\omega \mapsto \omega t$ by

$$\omega = (y_0, y_1, \dots) \mapsto (y_0 + t, y_1, \dots) \stackrel{\text{def}}{=} \omega t.$$

Clearly, these transformations preserve $\mathcal{B}_n$ for all n. Thus, they preserve the σ-algebra $\mathcal{B}_{\text{tail}}$ as well. We obtained the probabilistic space $(\Omega, \mathcal{B}_{\text{tail}}, P_{\text{tail}})$ on which the additive group $\mathbb{R}$ acts. (P_{tail} denotes the restriction of the measure P to $\mathcal{B}_{\text{tail}}$.) This is by definition a tail dynamical system; clearly, the measure P_{tail} is quasi-invariant under these transformations. Following the classical canons we consider the Hilbert space $\mathcal{H}_{\text{tail}} = L_2(\Omega, \mathcal{B}_{\text{tail}}, P_{\text{tail}})$ and the group of unitary operators defined by

$$(1) \qquad (U_t f)(\omega) = \sqrt{\frac{dP_{\text{tail}}(\omega t)}{dP_{\text{tail}}(\omega)}} f(\omega t), \quad f \in \mathcal{H}_{\text{tail}}, \ t \in \mathbb{R}^q.$$

The problem is to determine the spectral type of this group. We are able to settle this problem under some rather restrictive conditions; these are the following:

A) Suppose Y_k is finite for any $k > 0$.

B) Put $L_k = \max\{y_1 - y_2 : y_1 \in Y_k,\ y_2 \in Y_k\}$ and $l_k = \min\{y_1 - y_2 : y_1 \in Y_k,\ y_2 \in Y_k,\ y_1 \neq y_2\}$. Then we suppose that

$$l_k = L_1 + \dots L_{k-1} - M_0,\ \forall k > 0$$

for some M_0.

Assuming that these conditions A) and B) are satisfied, we define for any positive integer m the formal infinite product

$$R_m^\infty(\lambda) = \prod_{k=m}^{\infty} \left| \sum_{y \in Y_k} \sqrt{\mu_k}(y) e^{i(y,\lambda)} \right|^2, \tag{2}$$

where $\lambda \in \mathbb{R}^q$ and (y, λ) denotes the standard product in $\mathbb{R}^q$. It can be easily shown that under conditions A) and B) this product converges to some measure τ_m in the sense of distribution theory. Clearly, $\tau_1 \preceq \tau_2 \preceq \dots$, and we denote by τ the measure defined as $\tau = \sum_{k=1}^{\infty} \epsilon_k \tau_k$ where $\epsilon_k > 0$ and ϵ_k rapidly tends to 0; thus, τ is determined up to equivalence. Consider now the group of unitary operators

$$(V_t f)(\lambda) = e^{i(t,\lambda) f(\lambda)},\ f \in L_2(\mathbb{R}^q, \tau),\ t \in \mathbb{R}^q. \tag{3}$$

Theorem 1 *There exists a unitary operator* $K : \mathcal{H}_{tail} \to L_2(\mathbb{R}^q, \tau)$ *satisfying the equality*

$$K^{-1} U_t K = V_t,\ \forall t \in \mathbb{R}^q$$

(Thus the groups (1) and (3) are unitarily equivalent.)

3. The noncommutative case. Take G to be the group

$$\left\{ g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},\ a, b, c \in \mathbb{R} \right\}.$$

Similar considerations lead to noncommutative versions of products (2). This provides examples of tail dynamical systems with "time" G having singular spectra (in the dual $\hat{G}$).