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THE MULTIFRACTAL SPECTRUM OF RIEMANN'S FUNCTION $\sum_{n>0} \frac{\sin \pi n^2 x}{n^2}$

This is an account of a recent article by Stéphane Jaffard[1].
Consider the function

$$\varphi(x) = \sum_{n \geq 1} \frac{\sin \pi n^2 x}{n^2}.$$

If $x_0 \in \mathbb{R}$ and $\alpha > 0$, we say that $\varphi \in C^\alpha(x_0)$ if there exists a constant C and a polynomial P of degree less than or equal to α such that

$$\left|\varphi(x) - P(x)\right| \leq C \left|x - x_0\right|^\alpha.$$

Let us define three functions:

$$\alpha(x_0) = \sup\left\{\beta \,\middle|\, \varphi \in C^\beta(x_0)\right\},$$

$$d(\beta) = \dim\left\{x \,\middle|\, \alpha(x) = \beta\right\},$$

where dim stands for the Hausdorff dimension, and

$$\eta(p) = \sup\left\{s > 0 \,\middle|\, \varphi \in B_p^{s/p,\infty}\right\}.$$

where $B_p^{s/p,\infty}$ is a Besov space.

If $x \in \mathbb{R}\backslash\mathbb{Q}$, let p_n/q_n be the n^{th} convergent of the continued fraction expansion of x and define $\tau(x)$ to be the upper bound of τ's such that $\left|x - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^\tau}$ for infinitely many n's such that p_n and q_n be not both odd.

Theorem 1

1. *For* $x \in \mathbb{R} \setminus \mathbb{Q}$*, one has* $\alpha(x) = \dfrac{1}{2} + \dfrac{1}{2\tau(x)}$*,*

2. $d(\alpha) = 4\alpha - 2$ *if* $\alpha \in [1/2, 3/4]$*,* $d(3/2) = 0$*, and* $d(\alpha) = -\infty$ *(which means that the corresponding set is empty) otherwise,*

3. $d(\alpha) = \inf_p \big(\alpha p - \eta(p) + 1\big)$ *for* $\alpha \leq 3/4$*.*

Here are the broad lines of proofs:

Let $\psi(x) = \dfrac{1}{\pi(x+i)^2}$. Taking ψ as the mother wavelet, consider the wavelet transform of φ: for $b \in \mathbb{R}$ and $a > 0$,

$$C(a,b) = \frac{1}{a}\int_{-\infty}^{+\infty} \varphi(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)\,dt = \frac{i\pi a}{2}\left(\vartheta(b+ia) - 1\right),$$

where ϑ is the Jacobi function $\sum_{n\in\mathbb{Z}} \exp(i\pi n^2 z)$. The size of $C(a,b)$ as (a,b) goes to $(0,x_0)$ is closely related to $\alpha(x_0)$. Estimates of the ϑ Jacobi function near real rational points are deduced from the behaviour of ϑ near 0 or 1 by using the action of the ϑ modular group. Then the behaviour of $C(a,b)$ near $(0,x_0)$ follows from the approximation of x_0 by its continued fraction convergents. This gives (1).

The proof of assertion (2) uses theorems on the Hausdorff dimension of linear sets defined in terms of continued fraction expansions (see [2]).

Zalcwasser's estimates of the L^p norm of $\sum_{1\leq j\leq n} e^{i\pi j^2 x}$ allow the determination of the function η. The fact that d is the Legendre transform of η, which is expressed by saying that φ satisfies the multifractal formalism, is then checked by direct computation.

References

[1] S. Jaffard, *The Spectrum of Singularities of Riemann's Function*, preprint, CERMA, Feb. 1994.

[2] K. Falconer, Fractal Geometry, John Wiley and Sons Ltd., 1990.