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INFINITE CONFORMAL ITERATED FUNCTIONS SYSTEMS AND MEASURABILITY OF MEASURE AND DIMENSION FUNCTIONS

I discussed several uses of properties of measures and dimension. First, I discussed recent work of Pertti Mattila and myself examining the descriptive complexity of packing measure and dimension. We consider this complexity by examining the measurability properties of packing measure and dimension as a function from the space of compact sets into the extended real numbers.

Recall that the s-dimensional Hausdorff measure, H_s, is defined as follows:

$H_{s,\delta}(A) = \inf\{\sum d(E)^s | E \in \Gamma$ and Γ is a cover of A by sets with diameter less than $\delta\}$,

$H_s(A) = \lim_{\delta\to 0} H_{s,\delta}(A)$.

The s-dimensional packing measure, Π_s, unlike Hausdorff measure involves two limiting processes in its definition.

First, there is the s-dimensional prepacking measure P_s:

$P_{s,\delta}(A) = \sup\{\sum (2r_i)^s | B(x_i, r_i) \in \Gamma$ and $x_i \in A$, $r_i < \delta$ and the balls in Γ are disjoint$\}$,

$P_s(A) = \lim_{\delta\to 0} P_{s,\delta}(A)$.

Then $\Pi_s(A) = \inf\{\sum P_s(A_i) | \{A_i\}$ is a cover of $A\}$.

The Hausdorff and packing dimension functions are then defined as usual. I also use the notation $\overline{dim}_B$ and $\underline{dim}_B$ for the upper and lower "box counting" or Minkowski dimensions.

Several of these functions are Borel measurable.

Theorem 1 *Let X be a complete separable metric space and $K(X)$ the space of compact subsets of X with the Hausdorff metric. For each s and $\delta > 0$, the function $H_{s,\delta}$ is upper semicontinuous and $P_{s,\delta}$ is lower semicontinuous. For each s, the functions H_s and P_s are of Baire's class two. The functions dim_H, $\overline{dim}_B$, and $\underline{dim}_B$ are of Baire's class two.*

However, there is no simple definition of packing measure and dimension. I mean this in the sense that, unlike Hausdorff measure and dimension, these concepts cannot be defined in a Borel measurable manner. So, even if one gives a definition of these notions involving one limiting process, there is necessarily embedded in the definition some higher order quantification. On the other hand, the concepts still fall within the realm of reasonable sets for analysis.

Theorem 2 *Let $BA(X)$ be the σ-algebra generated by the analytic sets. For each s, the function Π_s is measurable with respect to $BA(X)$. The function dim_P is also measurable with respect to this algebra. However, if $X = [0,1]$, then neither of these functions is Borel measurable.*

I also discussed some recent work of Mariusz Urbanski and myself concerning the geometric measure theoretic properties of limit sets generated by iterating a system of infinitely many contracting conformal maps. Provided the system satisfies some regularity condition (expressed in terms of a zero of a pressure function), there is a "conformal" probability measure supported on the limit set. The pointwise scaling behaviour of this conformal measure governs the behaviour of the s-dimensional Hausdorff and packing measure on the limit set, where s is the dimension of the limit set. In general, s is the number where the pressure function has a zero.

As an application of these methods, I considered A, the residual set for Apollonian packing. This set is the limit set from iterating infinitely many conformal maps. If s is the Hausdorff dimension of A, then from the behaviour of the conformal measure on this set, it follows that $0 < H_s(A) < \infty$ and $\Pi_s(A) = \infty$, and $s = dim_P(A)$.

I also gave an example of a set of complex continued fractions studied by Richard Gardner and myself. This set can also be regarded as the limiting set of an infinite system of conformal maps. In this case, the pointwise scaling behaviour of the conformal measure yields $0 < \Pi_s(A) < \infty$ and $H_s(A) = 0$, and $1 < s < dim_H(A) = dim_P(A) < 2$.

References

[1] P. Mattila and R. D. Mauldin, *Measurability and measure and dimension functions*, (preprint).

[2] R. D. Mauldin and M. Urbanski, *Dimensions and measures in infinite iterated function systems*, (preprint).

[3] R. J. Gardner and R. D. Mauldin, *On the Hausdorff dimension of a set of complex continued fractions*, Illinois J. Math. **27** (1983), 334-344.