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ON SOME PROBLEMS OF FRACTIONAL DERIVATIVES

1 Basic Definitions

Definition 1 *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a continuous function. Let* $x \in]a,b], \alpha \in]0,1[$ *and* $n \in \mathcal{N}$ $(n > \alpha)$. *Following Riemann and Liouville, we define:*

1. ${}_aD_x^{-\alpha} f(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^x f(t)(x-t)^{\alpha-1} dt$, *to be the* α*-integral of* f. *And*

2. ${}_aD_x^{\alpha} f(x) = \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_a^x f(t)(x-t)^{n-\alpha-1} dt$, *as its* α*-derivative.*

We note the following properties:

- ${}_aD_x^{\alpha}$ is a linear operator whose definition is independent of the choice of n.
- If n is a positive integer then ${}_aD_x^{n} f(x) = \frac{d^n f(x)}{dx^n}$.
- For all positive reals α and β : ${}_aD_x^{-\beta} \left({}_aD_x^{-\alpha} f(x)\right) = {}_a D_x^{-(\alpha+\beta)} f(x)$.

Definition 2 *We note by:*

1. $\delta(f,x) = \sup\{\alpha$ *such that* ${}_aD_x^{\alpha} f(x)$ *exists*$\}$.
2. $\delta(f) = \sup_{x \in]a,b]} \{\delta(f,x)\}$.

We recall that the function f is said to be *hölderian* of *exponent* H, if $|f(x) - f(y)| \leq C\,|x-y|^H$, for some positive constant C.

It is *anti-hölderian* of *exponent* H, if $|f(x) - f(y)| \geq C_1\,|x-y|^H$, for some positive constant C_1.

Tricot [2] asks, among other questions, the following two:
If α is a non-integer, and ${}_aD_x^{\alpha} f(x)$ exists, then

1. Does ${}_aD_x^{\beta} f(x)$ exist for all $\beta \leq \alpha$?
2. Does ${}_bD_x^{\alpha} f(x)$ exist for all $b < x$, and $b \in \mathbb{R} \bigcup \{-\infty\}$?

2 Main Results

Problem 1 In 1927, Hardy and Littlewood [1] showed that a positive answer can be obtained when f is hölderian:

Theorem 1 *If f is hölderian of exponent H then ${}_aD_x^\alpha f(x)$ exists for every $\alpha < H$.*

Using Abel's criteria Tricot [2] proved:

Theorem 2 *Let $f(x) = \sum_0^\infty c_n e^{inx}$ where $x \in \mathbb{R}$, and $c_n \in \mathbb{C}$, assume that ${}_aD_x^\alpha f(x)$ exists, then for every $\beta \leq \alpha$ ${}_aD_x^\beta f(x)$ exists.*

One can prove the following:

Theorem 3 *Assume that $f(x) \geq 0$ and that for all $x \in [a, b]$ the integral $\int_a^x f(t)(x-t)^{-\alpha-1}dt$ converges. If ${}_aD_x^\alpha f(x)$ exists then so does ${}_aD_x^\beta f(x)$ for every $\beta \leq \alpha$.*

Theorem 4 *Assume that ${}_aD_x^\alpha f(x)$ exists and is bounded on every subinterval $[c, d]$ of $[a, b]$ with $c \neq a$, then for every $\beta \leq \alpha$ and $c \in]a, b]$, ${}_cD_x^\beta f(x)$ exists.*

Problem 2 A positive answer can, almost trivially, be obtained for the case where $b > -\infty$. As for $b = -\infty$, Tricot [2] showed:

Theorem 5 *If $\int_c^d f(t)dt$ is bounded for any $c, d \in \mathbb{R}$ and if ${}_aD_x^\alpha f(x)$ exists then so does ${}_{-\infty}D_x^\alpha f(x)$.*

These results may be used to show that: the Weierstrass function $W(x) = \sum_{n=0}^\infty b^{-nH}\cos(b^n x)$ (where $b > 1$ and $0 < H < 1$), verifies: $\delta(W) = H$; while the Riemann function $R(x) = \sum_{n=1}^\infty n^{-2}\sin(n^2 x)$, verifies: $\delta(R) = \frac{1}{2}$.

References

[1] G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals I*, Math. Z. **27**(1928), 565–606.

[2] C. Tricot, *Derivation fractionnaire et dimension fractal*, Tech. Report 1532, CRM-Université de Montréal, 1988.