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BOUNDED HARMONIC VARIATION AND THE GARSIA-SAWYER CLASS

Waterman's Test [3, Thm.1] includes all other tests that yield Dirichlet-Jordan conclusions. In particular, $GS \subset HBV$ where GS denotes the Garsia-Sawyer class defined in [1] first, and next extended to the non-continuous case by C. Goffman and D.Waterman [3, pp.109-111]. Unlike HBV, GS is not a linear space. Since Λ-variation furnishes a natural norm $\|f\|_\Lambda = |f(0)| + V_\Lambda(f)$ that makes ΛBV a Banach space, the question of whether or not the closure of GS in $\| \|_H$-norm is the whole space HBV was of some interest [5, Problem 3].

While investigating Cesàro summability of Fourier series, D. Waterman found it useful to define the class of functions continuous in Λ-variation (ΛBV_c) by $f \in \Lambda BV_c$ if and only if $\lim_n V_{\Lambda_{(n)}}(f) = 0$ [4]. Although $\Lambda BV_c \subset \Lambda BV$, the question about the exact relationship between ΛBV_c and ΛBV was around for almost 20 years, and there were a number of partially successful attempts to answer it. In 1978 R. Fleissner and J. Foran showed that ΛBV_c might be a proper subclass of ΛBV. A sufficient condition for ΛBV_c to be a proper subset of ΛBV was given by a A. I. Sablin, in 1985. In the same year in China, G. Shao published a sufficient condition for the two classes to be equal. Two years later, Sablin announced another sufficient condition for the equality $\Lambda BV = \Lambda BV_c$. Eventually, it turned out that

Theorem 1 $\Lambda BV = \Lambda BV_c$ *if and only if* $\limsup_n \frac{\sum_1^{2n} \frac{1}{\lambda_i}}{\sum_1^{n} \frac{1}{\lambda_i}} < 2$.

In order to explain the numerous nice properties that functions continuous in Λ-variation enjoy, we need a new definition.

Definition 1 *A function f is said to be Λ-absolutely continuous (in symbols $f \in \Lambda AC$) if for every $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\sum_i \frac{|b_i - a_i|}{\lambda_i} < \delta \quad \Rightarrow \quad \sum_i \frac{|f(b_i) - f(a_i)|}{\lambda_i} < \epsilon$$

for every collection $\{[a_i, b_i]\}$ *of nonoverlapping intervals.*

Theorem 2 *If* $\lambda_i \to \infty$, *then* $\Lambda AC = C \cap \Lambda BV_c$.

Theorem 3 [2] $(\Lambda AC, \|\,\|_\Lambda)$ *is separable. The set of piecewise linear continuous functions is dense there.*

Definition 2 *A regulated function* f *is said to have identical points of discontinuity with a regulated function* g *in case* $f(t+) = f(t)$ *iff* $g(t+) = g(t)$, *and* $f(t-) + f(t)$ *iff* $g(t-) = g(t)$ *for every* $t \in [0,1]$.

If h is a strictly increasing function has identical points of discontinuity with a regulated function f , then the function $f \circ h^{-1}$ possesses a unique continuous extension to $\overline{h([0,1])}$ that we will again denote by $f \circ h^{-1}$, and that can be further extended to a continuous function $\overline{f \circ h^{-1}}$ on the entire interval $[h(0), h(1)]$ by requiring that it is linear on the closure of each interval complementary to $\overline{h([0,1])}$.

Theorem 4 *Let* $\lambda_i \to \infty$, f *be a regulated function and let* h *be any strictly monotone function having the same points of discountinuity as* f. *Then* $f \in \Lambda BV_c$ *if and only if* $\overline{f \circ h^{-1}} \in \Lambda AC$.

This theorem combined with the previous result on separability of $(\Lambda AC, \|\,\|_\Lambda)$ easily furnishes the following fact.

Theorem 5 *BV is a dense subset of* $(\Lambda BV_c, \|\,\|_\Lambda)$ *if* $\lambda_i \to \infty$.

Since $BV \subset GS$, we get by Theorem 1

Corollary 1 *The Garsia-Sawyer class is a dense subset of* $(HBV, \|\,\|_H)$.

References

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