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A_p-WEIGHTS AND RELATED TOPICS

Let Γ be a smooth curve in a complex plane $\mathbb{C}$, and let ρ be a measurable a.e. positive function defined on Γ. A *weighted L_p-space* is defined as

$$L_{p,\rho}(\Gamma) = \{f : \rho f \in L_p(\Gamma)\},$$

and $L_{p,\rho}$-norm of f is, by definition, an L_p-norm of ρf.

We say that $\rho \in W_p(\Gamma)$ if the Cauchy integral operator S, acting according to the formula

$$(S\phi)(t) = \frac{1}{2\pi i}\int_\Gamma \phi(\tau)\frac{d\tau}{\tau - t},$$

is bounded in $L_{p,\rho}$-norm. It is well known ([4], see also [3]) that $\rho \in W_p(\Gamma)$ if and only if ρ^p is an A_p-weight, that is,

$$\sup \frac{1}{|I|}\left(\int_{|I|} \rho^p(t)\,|dt|\right)^{1/p}\left(\int_{|I|} \rho^{-q}(t)\,|dt|\right)^{1/q} < \infty,$$

where sup is taken against all arcs I of Γ, $|I|$ stands for the length of I, $p \in (1,\infty)$, and $1/p + 1/q = 1$.

For any $t \in \Gamma$, denote

$$J_t(\rho) = \{\alpha \in \mathbb{R} : |\tau - t|^\alpha \rho(\tau) \in W_p(\Gamma)\}.$$

It was shown in [6] that $J_t(\rho)$ is an open interval $(-\nu_-(t), 1 - \nu_+(t))$, where $0 < \nu_-(t) \le \nu_+(t) < 1$, and that numbers $\nu_\pm(t)$ play a crucial role in the theory of Toeplitz and singular integral operators on $L_{p,\rho}$. In particular, an operator $R_G = P_+ + GP_-$ with piecewise continuous symbol G and $P_\pm = 1/2(I \pm S)$ is Fredholm on $L_{p,\rho}$ if and only if $G(t \pm 0) \ne 0$ and

$$\frac{1}{2\pi}\arg\frac{G(t+0)}{G(t-0)} \notin [\nu_-(t), \nu_+(t)] \text{ for all } t \in \Gamma. \tag{1}$$

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According to this result, the weights ρ for which

$$\forall t \in \Gamma \quad \nu_+(t) = \nu_-(t) \tag{2}$$

are distinguished by the property that essential spectra $\sigma_{ess}(R_G)$ of all acting on $L_{p,\rho}$ operators R_G with piecewise continuous symbols are one-dimensional. A classical example of weights satisfying (2) is delivered by *power* weights

$$\rho(\tau) = \prod_{j=1}^{N} |\tau - t_j|^{\beta_j},$$

for which a straightforward calculation shows that

$$\nu_+(t) = \nu_-(t) = \frac{1}{p} + \beta(t), \text{ where } \beta(t) = \begin{cases} \beta_j & \text{if } t = t_j,\ j = 1, \dots, N \\ 0 & \text{otherwise.} \end{cases}$$

For such weights, description of $\sigma_{ess}(R_G)$ was obtained in [2], see also [3].

We will say that ρ is a *power-like* weight if (2) holds. It was shown in [1] that weights of the form $|x|^\beta v(\log|x|)$ $(-1/p < \beta < 1/q,\ v \in W_p(\mathbb{R}))$, introduced by Rooney [5], are power-like, but a complete description of all power-like weights remains an open problem.

Another open problem, brought to author's attention by S. Grudskii, can be stated as follows. Let (for simplicity) Γ be the unit circle $\mathbb{T}$, and let u be an inner function of infinite degree. Is it true that

$$\rho \in W_p(\mathbb{T}) \Rightarrow \rho \circ u \in W_p(\mathbb{T}) \quad ? \tag{3}$$

A positive answer to this question would allow to develop a theory of Toeplitz and singular integral operators with infinite defect numbers and closed range.

So far, we were able to prove only that

$$\rho \in W_p(\mathbb{T}) \cap W_q(\mathbb{T}) \Rightarrow \rho \circ u \in W_p(\mathbb{T}) \cap W_q(\mathbb{T}).$$

Of course, it means that the answer to (3) is positive for $p = 2$.

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