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ω-LIMIT SETS AND CONTINUOUS FUNCTIONS WITH CONTROLLED GROWTH

Let C denote the class of continuous self-maps of an interval I. For $f \in C$ and $n \in \mathbb{N}$, we adopt the notation $f^1 = f$, $f^{n+1} = f \circ f^n$.

Definition 1 *A set $E \subseteq I$ is an ω-limit set for $f \in C$ if there exists $x \in I$ such that the cluster set of the sequence $\{f^n(x)\}$ equals E.*

Definition 2 *Let $f \in C$, and $\sigma : I \to \mathbb{R}$ be continuous and nondecreasing with $\sigma(0) = 0$. We say $f \in C_L(\sigma)$ if for any $x \in I$ there exists a relative neighborhood $U_x \subseteq I$ of x such that for any $y \in U_x$, $|f(x) - f(y)| \leq \sigma(|x - y|)$. Let Lipschitz (σ) be the set $\{f \in C_L(k\sigma) : k \in \mathbb{N}\}$.*

Necessary and sufficient conditions for a set E to be an ω-limit set for some $f \in C$ have been obtained in [1] and [2]. In particular, every Cantor set is such an ω-limit set. In [3] we were able to show that the cleavage between those sets that are ω-limit sets for continuous functions and those sets that are ω-limit sets for Lipschitz functions is significant. In fact, most Cantor sets are not ω-limit sets for Lipschitz functions.

It is natural, then, to ask whether these most recent results apply to other moduli of continuity as well. As before, we study the question in the context of continuous maps of a Cantor set onto itself, since any continuous function is strongly invariant on its ω-limit sets.

Let $\mathcal{K}$ denote the class of nonempty closed subsets of I. We furnish $\mathcal{K}$ with the Hausdorff metric, producing a compact metric space in which the Cantor sets form a dense G_δ.

Theorem 1 *Let $\sigma : I \to \mathbb{R}$ be continuous and non-decreasing with $\sigma(0) = 0$. There exists a residual subset $\mathcal{E}(\sigma)$ of $\mathcal{K}$ such that if $E \in \mathcal{E}(\sigma)$, and $f : E \to E$ is continuous and not the identity on any portion of E, then the following statements are valid:*

1. *There exists a Cantor set $K \subseteq E$ such that f is not Lipschitz (σ) on any portion of K.*

2. *$f(K) = E$.*

3. *If $P = (a,b) \cap E$ is a portion of E contiguous to K, then $f(P)$ is nowhere dense in E.*

4. *If f is Lipschitz (σ) on a subset C of E, then $f(C)$ is nowhere dense in E.*

5. *f is nondifferentiable on a dense G_δ of K.*

6. *f maps the set of points of differentiability in E onto a first category subset of E.*

Since f is not the identity on any portion of E whenever E is an ω-limit set of f, we have that most closed sets are not ω-limit sets for Lipschitz (σ) functions.

We can also consider the class of sets $\mathcal{K}_c$ composed of those closed subsets of I with Lebesgue measure at least c. By furnishing $\mathcal{K}_c$ with the Hausdorff metric, we again get a compact metric space. Moreover, we are also able to obtain the same results as those in Theorem 1, though the proofs rest on entirely different ideas.

Our final theorem shows that even though the typical nowhere dense compact set is not an ω-limit set for a continuous function that meets a specific local modulus of continuity condition, all nowhere dense compact sets do have homeomorphic copies that are ω-limit sets for differentiable Lipschitz functions.

Theorem 2 *Every nowhere dense compact set M contained in I has a homeomorphic copy M^* in I that is a homoclinic attractor for a differentiable Lipschitz function.*

References

[1] S. J. Agronsky, A. M. Bruckner, J. G. Ceder and T. L. Pearson, *The structure of for continuous functions*, Real Anal. Ex. **15** (1990), 483-510.

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[3] A. M. Bruckner and T. H. Steele, *The Lipschitz structure of continuous self-maps of generic compact sets*, J. Math. Anal. and Appl., to appear.