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ω-LIMIT SETS FOR CERTAIN CLASSES OF FUNCTIONS

Let $\mathcal{C}$ denote the class of continuous self-maps of a closed interval I. For $f \in \mathcal{C}$ let $f^1 = f$ and let $f^{n+1} = f \circ f^n$ for each $n \in \mathbb{N}$.

Definition 1 *A set $E \subset I$ is an* ω-limit set *for $f \in \mathcal{C}$ if there exists $x \in I$ such that the cluster set of the sequence $\{f^n(x)\}$ equals E. We then write $E = \omega(x, f)$.*

Some recent work has focused on characterizing the ω-limit sets for functions in certain subclasses $\mathcal{F}$ of $\mathcal{C}$. The class $\mathcal{F}$ might be a class of functions exhibiting some form of nonchaos, or it might be a class exhibiting some form of smoothness.

For $\mathcal{F} = \mathcal{C}$, we have the following theorem.

Theorem 1 [1],[3] *A necessary and sufficient condition that a set $E \subset I$ be an ω-limit set for some $f \in \mathcal{C}$ is that E be either a nonempty nowhere dense closed set or that E be a finite union of closed intervals.*

If we restrict ourselves to functions nonchaotic in certain senses, we obtain two more theorems.

Theorem 2 [4] *A necessary and sufficient condition that a set $E \subset I$ be an ω-limit set for some continuous function of zero topological entropy is that E be either a finite set of cardinality 2^n, for some $n \in \mathbb{N}$, or that E be of the form $E = P \cup C$ where P is a Cantor set and C is either empty or satisfies the conditions below:*

(1) $\overline{C} = E$,

(2) $\text{card}(C \cap J) \leq 2$ *for every interval J contiguous to P, and*

*Professor Bruckner was unable to attend the symposium. T. H. Steele presented a summary of this work.

(3) $\operatorname{card}(C \cap J) \leq 1$ *if* $\sup J \leq \inf P$ *or* $\inf J \geq \sup P$.

A more restrictive form of nonchaos limits the possible ω-limit sets further. Let $\mathcal{K}$ denote the nonempty compact sets of I furnished with the Hausdorff metric. Define $\omega_f : I \to \mathcal{K}$ by $\omega_f(x) = \omega(x, f)$. In [2] one finds that ω_f is always in the second Baire class, and that when ω_f is in the first Baire class, f has zero topological entropy, and therefore f exhibits a form of nonchaos.

Theorem 3 [2] *A necessary and sufficient condition that a set $E \subset I$ be an ω-limit set for some $f \in \mathcal{C}$ with ω_f in the first class of Baire is that E be either a nonempty finite set of cardinality 2^n, for some $n \in \mathbb{N}$, or a Cantor set.*

Theorems 2 and 3 characterize ω-limit sets for the two classes of functions. It is also true that a function is in the class if and only if all its ω-limit sets are of the type stated.

A still more restrictive notion of nonchaos is the requirement that f possess no scrambled sets. This class has the same family of ω-limit sets as the class in Theorem 3, so this family cannot be characterized by its ω-limit sets.

Incidentally, for the still more restrictive notion of nonchaos that ω_f be in O'Malley's class $\mathcal{B}_1^*$, the possible ω-limit sets are just the finite sets of cardinality 2^n for some $n \in \mathbb{N}$.

Less seems to be known about the possible ω-limit sets for classes of functions restricted by smoothness conditions. It is known [5] that there exists a residual subspace $\mathcal{K}^*$ of the compact metric space $\mathcal{K}$, such that if $K \in \mathcal{K}^*$, then K is not an ω-limit set for any Lipschitz function or for any differentiable function. In fact, for such a set K, if $K = \omega(x, f)$ for some f in $\mathcal{C}$, then f must map any set on which it satisfies a Lipschitz condition onto a nowhere dense subset of K.

This raises several questions:

1. How can one characterize ω-limit sets for the Lipschitz functions?
2. What about Lipschitz functions that possess a form of nonchaos?
3. What is the situation for other forms of smoothness, e.g. for $f \in \mathcal{C}^{\backslash}$ for some n, $1 \leq n \leq \infty$, or that f satisfy a certain modulus of continuity?
4. How can one characterize ω-limit sets *up to homeomorphism* for such classes of functions?

These questions seem not to have been fully answered. Some partial answers to these questions, as well as complete answers to some related questions, have recently been obtained by T. H. Steele [6].

References

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