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Loren D. Pitt, Department of Mathematics, University of Virginia, Charlottesville, VA 22903 *

DISTORTION THEORY FOR FUNCTIONS IN A ZYGMUND SPACE Λ^*

The Zygmund space Λ^* consists of continuous real valued functions f of period 1 with $||\Delta_{2,h}f||_\infty = O(h)$ as $h \downarrow 0$, where $\Delta_{2,h}f(x) = \frac{1}{2}[f(x+h) + f(x-h)] - f(x)$. For f in Λ^* the best possible uniform modulus of continuity is $O(\delta \log(1/\delta))$. Examples of nowhere differentiable functions in Λ^* are the Weierstrass function $w(x) = \Sigma 2^{-k}\cos(2\pi 2^k x)$ and the related Kahane function $k(x) = \Sigma 2^{-k} S(2^k x)$, where $S(x)$ is the piecewise linear function with nodes at $n/4, \sin(n\pi/2)/4$.

Three distinct questions were discussed concerning nondifferentiable functions f in Λ^*.

Q1. Is the image of Lebesgue measure $\mu_f(A) = |\{x : f(x) \in A\}|$ a singular measure?

The answer is unknown even for the Weierstrass function $w(x)$, but μ_f is singular when $||\Delta_{2,h}f||_\infty = o(h)$, and for general nondifferentiable f in Λ^*, $\mu_{f+\beta x}$ is singular for a.a. β. See [1] and [3]. The proofs rest on the fact that $||f - f_n||_\infty = O(2^{-n})$, where $f_n(x)$ is the piecewise linear function with nodes at $\{(k2^{-n}, f(h2^{-n})\}$.

Q2. Is there a generic local modulus of continuity $w_0(\delta)$ which is smaller than $\delta \log(1/\delta)$, and which is satisfied for a.a. x?

The answer is yes with $w_0(\delta) = \delta\sqrt{\log(1/\delta)\log\log\log(1/\delta)}$. The derivation depends on the rate of growth of the sequence $\{f_n'(x)\}$ of derivatives of the function $f_n(x)$ mentioned in Q1. The key is that $\{f_n'(x)\}$ is a martingale sequence on $[0,1]$, and the result follows from the martingale law of the iterated logarithm (LIL). See [1] and [4].

Q3. In analogy with multi-fractal decompositions, is it possible to analyze the set of exceptional points where the LIL fails for $\{f_n'(x)\}$.

The answer is yes. Refined information such as dimensions and integral tests are available for both slow points and fast points. See [2].

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References

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