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THE RANGE OF A SYMMETRIC DERIVATIVE

The range of ordinary derivatives is easy enough to sort out. If f is continuous and has a derivative everywhere, even allowing infinite values, then f' has the Darboux property. Thus the range of f' must be an interval or a single point.

For symmetric derivatives these questions are rather more delicate. For example the continuous function $f(x) = |x|$ is everywhere symmetrically differentiable and its symmetric derivative assumes just the three values 0, 1 and -1. The Cantor function is also continuous and everywhere symmetrically differentiable and its symmetric derivative assumes just the two values 0 and $+\infty$. Buczolich and Laczkovich [1, Theorem 5.1, p. 359] show that there is no possibility of two *finite* values.

Our purpose in this short article is to present an entirely elementary proof of this theorem. This is largely to bring this theorem to the attention of those collectors of symmetric arcana who otherwise might miss this result, buried as it is in a paper mainly devoted to the structure of certain Borel measures.

The proof we give here uses only three of the most immediate properties of symmetric derivatives. A continuous function with a nonnegative symmetric derivative is nondecreasing; this was first proved by Khintchine [2] but requires nothing more than familiar nineteenth century arguments. At any point the symmetric derivative is clearly the average of the two one-sided derivatives when they exist; in fact if any two of $\mathrm{SD}f(x)$, $f'_+(x)$ and $f'_-(x)$ exist so does the other and $\mathrm{SD}f(x) = \frac{1}{2}(f'_+(x) + f'_-(x))$. Finally any symmetric derivative of a continuous function is evidently in the first Baire class. From these facts we construct our proof avoiding some of the heavier artillery called to the front in [1].

THEOREM 1 **(Buczolich–Laczkovich)** *There is no symmetrically differentiable function whose symmetric derivative assumes just two finite values.*

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PROOF. Our first observation is that the theorem can be reduced to showing that there is no *continuous* function with this property. This exploits some work of Larson [3]; he shows that if a function g exists with a bounded, symmetric derivative everywhere then there is a continuous function f for which $\mathrm{SD}f(x) = \mathrm{SD}g(x)$ everywhere.

We assume then, contrary to the theorem, that there is a continuous, symmetrically differentiable function f whose symmetric derivative assumes only the two distinct values α and β, $\alpha < \beta$. ¿From the fact that $\alpha \le \mathrm{SD}f(x) \le \beta$ the monotonicity theorem shows that both $f(x) - \alpha x$ and $\beta x - f(x)$ are nondecreasing.

Since $\mathrm{SD}f(x)$ is Baire 1 there are points of continuity of $\mathrm{SD}f(x)$ in every interval. But at a point of continuity there must be an interval in which $\mathrm{SD}f(x)$ assumes only the value α or the value β. In such an interval the monotonicity theorem, applied once again, shows that f is linear with slope α or β. Thus there is a maximal open set G so that in every component of G the function f is linear with slope α or β.

Let P denote the complement of G. P can have no isolated points. For if $b \in P$ and (a, b), $(b, c) \subset G$ then f is linear with slope α or β in each interval $[a, b]$, $[b, c]$. If the slope is the same in the two intervals then f is linear on $[a, c]$ which contradicts the maximality of G. If the slope is different in the two intervals then $\mathrm{SD}f(b) = \frac{1}{2}(\alpha + \beta)$ and this value is not allowed for the symmetric derivative.

If fact P must be empty. If not then P is perfect and, again using the fact that $\mathrm{SD}f(x)$ is Baire 1, there is a point of continuity of $\mathrm{SD}f(x)$ relative to P. Thus there must be a nonempty portion $P \cap (a, b)$ so that either $\mathrm{SD}f(x) = \alpha$ for all $x \in P \cap (a, b)$ or $\mathrm{SD}f(x) = \beta$ for all $x \in P \cap (a, b)$.

Let us suppose the latter case; the argument for the former is similar. Consider some interval $[c, d]$ contiguous to P in (a, b). In the interval $[c, d]$ the function f is linear with slope α or β. Since $\mathrm{SD}f(c) = \beta$ and $f'_+(c)$ is either α or β it follows that $f'_-(c)$ exists too. But, since $f(x) - \alpha x$ and $\beta x - f(x)$ are nondecreasing, $\alpha \le f'_-(c) \le \beta$. This shows that

$$f'_+(c) = 2\,\mathrm{SD}f(c) - f'_-(c) \ge 2\beta - \beta = \beta$$

and so f cannot have slope α in $[c, d]$. Thus in this case in every interval contiguous to P in (a, b) the function f is linear with slope β. This means that $\mathrm{SD}f(x) = \beta$ for all $x \in (a, b)$ and hence f is linear in (a, b) which contradicts the fact that the portion $P \cap (a, b)$ is nonempty.

We can conclude that P must be empty and so we see that f can only be linear. This contradicts the fact that its symmetric derivative assumes two values and the conclusion of the theorem follows.

A symmetric derivative may, as already stated, assume three distinct finite values. Indeed let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq \beta$. Then there is a continuous, symmetrically differentiable function f such that its symmetric derivative assumes just the three finite values α, β and $\frac{1}{2}(\alpha+\beta)$. (Simply bend the example $f(x) = |x|$ into the right shape.) Using the arguments of Theorem 1, we can show that no other configuration is possible.

THEOREM 2 *Let α, β, $\gamma \in \mathbb{R}$ with $\alpha < \gamma < \beta$ and $\gamma \neq \frac{1}{2}(\alpha + \beta)$. Then there is no symmetrically differentiable function whose symmetric derivative assumes just the three values α, β and γ.*

PROOF. As in the preceding proof we need only show that there is no continuous function f with this property. If there is then, as before, both $f(x) - \alpha x$ and $\beta x - f(x)$ are nondecreasing.

We show that this cannot happen. Since $\mathrm{SD}f(x)$ is Baire 1 there are points of continuity of $\mathrm{SD}f(x)$ in every interval. But at a point of continuity there must be an interval in which $\mathrm{SD}f(x)$ assumes only the value α, β or γ; in such an interval f is linear with slope α, β or γ. Thus there is a maximal open set G so that in every component of G the function f is linear with slope α, β or γ.

Let P denote the complement of G. Exactly as before P can have no isolated points. If P is not empty then P is perfect and, yet again using the fact that $\mathrm{SD}f(x)$ is Baire 1, there is a point of continuity of $\mathrm{SD}f(x)$ relative to P. Thus there must be a nonempty portion $P \cap (a, b)$ so that $\mathrm{SD}f(x)$ assumes just one of the three values α, β or γ for all $x \in P \cap (a, b)$.

Let us suppose the value assumed is α. Consider some interval $[c, d]$ contiguous to P in (a, b). In the interval $[c, d]$ the function f is linear with slope α, β or γ. But, exactly as argued in the proof of Theorem 1, it cannot have slope β. This means that in the entire interval (a, b) the symmetric derivative assumes only the two values α or γ. But by Theorem 1 itself no function can exist with just two values for its symmetric derivative in an interval. Thus this case cannot occur.

In the same way we may suppose that the value assumed is β and again obtain a contradiction.

Thus we arrive now at the case that $\mathrm{SD}f(x)$ assumes just the value γ for all $x \in P \cap (a, b)$. We may suppose, without loss of generality that $\gamma > \frac{1}{2}(\alpha + \beta)$. Consider some interval $[c, d]$ contiguous to P in (a, b). In the interval $[c, d]$ the function f is linear with slope α, β or γ.

Since $\mathrm{SD}f(c) = \gamma$ and $f'_+(c)$ is either α, β or γ it follows that $f'_-(c)$ exists too. But, since $f(x) - \alpha x$ and $\beta x - f(x)$ are nondecreasing, $\alpha \leq f'_-(c) \leq \beta$.

This shows that

$$f'_+(c) = 2\,\mathrm{SD}f(c) - f'_-(c) \geq 2\gamma - \beta > \alpha$$

and so f cannot have slope α in $[c,d]$. Thus in this case in every interval contiguous to P in (a,b) the function f is linear with slope β or γ. This means that in the entire interval (a,b) the symmetric derivative assumes only the two values β or γ. Again by Theorem 1 no function can exist with just two values for its symmetric derivative in an interval. Thus this case cannot occur.

As we have eliminated all possible cases we see that, as before, P must be empty so that f can only be linear; this contradicts the fact that its symmetric derivative assumes three distinct values.

Evidently one might continue in this fashion asking for further conditions on the possible disposition of a symmetric derivative whose range is finite. I doubt, however, many readers could tolerate much more and few surprises are left in any case.

References

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