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BAIRE ONE FUNCTIONS AND PERFECT SETS

A function is a Baire class one function if it is the pointwise limit of a sequence of continuous functions. The following theorem provides two equivalent and quite useful characterizations of these functions.

Theorem *Let $f : [a,b] \to R$. The following are equivalent:*

1. *The function f is a Baire class one function.*
2. *For each real number r, the sets $\{x \in [a,b] : f(x) < r\}$ and $\{x \in [a,b] : f(x) > r\}$ are F_σ sets.*
3. *For each perfect set $P \subset [a,b]$, the function $f|_P$ (the restriction of f to P) has at least one point of continuity in P.*

The proof of this theorem can be found in several standard texts. See for instance Natanson [2]. The focus here will be on the fact that (3) implies (2). The typical proof of this implication uses transfinite numbers. We offer here a proof that avoids the use of transfinite numbers. The key result is the following lemma which is due to Romanovski [3]. A proof of this lemma can also be found in Gordon [1].

Romanovski's Lemma *Let $\mathcal{F}$ be a family of open intervals in (a,b) and suppose that $\mathcal{F}$ has the following properties:*

1. *If (α,β) and (β,γ) belong to $\mathcal{F}$, then (α,γ) belongs to $\mathcal{F}$.*
2. *If (α,β) belongs to $\mathcal{F}$, then every open interval in (α,β) belongs to $\mathcal{F}$.*
3. *If (α,β) belongs to $\mathcal{F}$ for every interval $[\alpha,\beta] \subset (c,d)$, then (c,d) belongs to $\mathcal{F}$.*

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4. *If all of the intervals contiguous to the perfect set $E \subset [a,b]$ belong to $\mathcal{F}$, then there exists an interval I in $\mathcal{F}$ such that $I \cap E \neq \emptyset$.*

Then $\mathcal{F}$ contains the interval (a,b).

PROOF OF (3) $\Rightarrow$ (2): The first step is to prove the following: if r and s are arbitrary real numbers with $r < s$, then there exist disjoint F_σ sets A and B such that $[a,b] = A \cup B$, $A \subset \{x \in [a,b] : f(x) > r\}$, and $B \subset \{x \in [a,b] : f(x) < s\}$. Let $\mathcal{F}$ be the collection of all open intervals in (a,b) that have such a decomposition. We will verify that $\mathcal{F}$ satisfies the four conditions of Romanovski's Lemma. It will then follow that (a,b) has the required decomposition and hence $[a,b]$ as well.

Suppose that (α,β) and (β,γ) belong to $\mathcal{F}$. Let $(\alpha,\beta) = C_r \cup C_s$ and $(\beta,\gamma) = D_r \cup D_s$ be the corresponding decompositions. Suppose that $f(\beta) > r$; the case $f(\beta) < s$ is similar. Then

$$(\alpha,\gamma) = (C_r \cup D_r \cup \{\beta\}) \cup (C_s \cup D_s)$$

is an appropriate decomposition of (α,γ). Hence $(\alpha,\gamma) \in \mathcal{F}$. If $(u,v) \subset (\alpha,\beta)$, then

$$(u,v) = (C_r \cap (u,v)) \cup (C_s \cap (u,v))$$

is an appropriate decomposition of (u,v). (The intersection of two F_σ sets is an F_σ set.) Hence $(u,v) \in \mathcal{F}$. This shows that $\mathcal{F}$ satisfies conditions (1) and (2).

Now suppose that (α,β) belongs to $\mathcal{F}$ for every interval $[\alpha,\beta] \subset (c,d)$. Let u be the midpoint of (c,d). We will prove that $(u,d) \in \mathcal{F}$. The proof that $(c,u) \in \mathcal{F}$ is similar, then $(c,d) \in \mathcal{F}$ by condition (1). Let $\{c_n\}$ be an increasing sequence in (u,d) and let $c_0 = u$. For each n, let $(c_{n-1},c_n) = A_n \cup B_n$ be an appropriate decomposition of (c_{n-1},c_n). Define

$$\pi_r = \{n \in Z^+ : f(c_n) > r\} \quad \text{and} \quad \pi_s = \{n \in Z^+ - \pi_r : f(c_n) < s\}.$$

Since a countable union of F_σ sets is still an F_σ set,

$$(u,d) = \left(\bigcup_{n=1}^{\infty} A_n \cup \{c_n : n \in \pi_r\}\right) \cup \left(\bigcup_{n=1}^{\infty} B_n \cup \{c_n : n \in \pi_s\}\right)$$

is an appropriate decomposition of (u,d). This shows that $\mathcal{F}$ satisfies condition (3).

Finally, suppose that all of the intervals contiguous to the perfect set $E \subset [a,b]$ belong to $\mathcal{F}$. By hypothesis, there exists a point $z \in E$ such that $f|_E$ is continuous at z. Suppose for the sake of definiteness that $f(z) < s$. Now there exists an interval $[c,d]$ such that $c,d \in E$, $E \cap (c,d) \neq \emptyset$, $z \in [c,d]$, and $f(x) < s$ for all $x \in E \cap [c,d]$. Let $[c,d] - E = \bigcup_{n=1}^{\infty}(c_n, d_n)$ and let $(c_n, d_n) = A_n \cup B_n$ be an appropriate decomposition of (c_n, d_n) for each n. Since $E \cap (c,d)$ is an F_σ set,

$$(c,d) = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} B_n \cup (E \cap (c,d))\right)$$

is an appropriate decomposition of (c,d). Hence $\mathcal{F}$ satisfies condition (4) of Romanovski's Lemma.

Now let r be an arbitrary real number and let $\{s_n\}$ be a decreasing sequence of real numbers that converges to r. By the above result, for each n there exist disjoint F_σ sets A_n and B_n such that $[a,b] = A_n \cup B_n$,

$$A_n \subset \{x \in [a,b] : f(x) > r\}, \quad \text{and} \quad B_n \subset \{x \in [a,b] : f(x) < s_n\}.$$

It is easy to verify that $\{x \in [a,b] : f(x) > r\} = \bigcup_{n=1}^{\infty} A_n$ and is therefore an F_σ set. Similarly, the set $\{x \in [a,b] : f(x) < r\}$ is an F_σ set. This completes the proof.

References

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