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ALGEBRA GENERATED BY NON-DEGENERATE DERIVATIVES

Abstract

In this paper we prove that each Baire one function $u : \mathbb{R}^m \to \mathbb{R}$ can be written as $u = f \cdot g + h$, where $f, g, h : \mathbb{R}^m \to \mathbb{R}$ are non-degenerate derivatives (both notions with respect to the ordinary differentiation basis).

In 1982 D. Preiss proved the following theorem [6].

Theorem 1 *Whenever $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of the first class there are derivatives $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $u = f \cdot g + h$. Moreover one can find such a representation that g is bounded and h is Lebesgue and in case u is bounded such that f and h are also bounded.*

The generalization of this theorem for derivatives of interval functions (with respect to the ordinary differentiation basis) was proved in 1989 by R. Carrese [2]. However, it is well known (and easy to prove) that derivatives needn't be non-degenerate everywhere. In this paper I prove that each Baire one function $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ can be written as $u = f \cdot g + h$, where $f, g, h : \mathbb{R}^m \longrightarrow \mathbb{R}$ are non-degenerate derivatives (both notions with respect to the ordinary differentiation basis). In the proof I use the Preiss's method.

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$, the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{N}$ and the set of rationals by $\mathbb{Q}$. To the end of this article m is a fixed positive integer. The word function means mapping from $\mathbb{R}^m$ into $\mathbb{R}$ unless otherwise explicitly stated. The words measure, almost everywhere (a.e.), summable etc. refer to

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the Lebesgue measure and integral in $\mathbb{R}^m$. We denote by $a \vee b$ ($a \wedge b$) not smaller (not greater) of real numbers a and b. The Euclidean metric in $\mathbb{R}^m$ will be denoted by ϱ. For every set $A \subset \mathbb{R}^m$, let $diam\, A$ be its diameter (i.e. $diam\, A = \sup\{\varrho(x,y) : x,y \in A\}$), χ_A its characteristic function and $|A|$ its outer Lebesgue measure. Symbol $\int_A f$ will always mean the Lebesgue integral. We say that f is a Baire one function, if it is a pointwise limit of some sequence of continuous functions. By $||f||$ we denote the sup norm of a function f (i.e. $||f|| = \sup\{|f(t)| : t \in \mathbb{R}^m\}$). Finally, the oscillation of a function f on a non-void set $A \subset \mathbb{R}^m$ will be denoted by $\omega(f, A)$ (i.e. $\omega(f,A) = \sup\{|f(x) - f(y)| : x,y \in A\}$).

The word *interval* (*cube*) will always mean non-degenerate compact interval (cube) in $\mathbb{R}^m$, i.e. Cartesian product of m non-degenerate compact intervals (compact intervals of equal length) in $\mathbb{R}$. We denote by Γ the family of all intervals.

Let $n \in \mathbb{N}$. We say that I is a *basic cube of order n*, if

$$I = \left[\frac{k_1}{2^n}, \frac{k_1+1}{2^n}\right] \times \ldots \times \left[\frac{k_m}{2^n}, \frac{k_m+1}{2^n}\right]$$

for some $k_1, \ldots, k_m \in \mathbb{Z}$. The family of all basic cubes of order n will be denoted by Γ_n. Elements of $\bigcup_{n=1}^{\infty} \Gamma_n$ will be called simply *basic cubes.*

Remark. Observe that for any two basic cubes I and J, either I and J do not overlap (i.e. $I \cap J \notin \Gamma$), or $I \subset J$, or $J \subset I$.

The following lemma is a slightly modified version of Lemma 2.1 of [5].

Lemma 2 *Let $A \subset \mathbb{R}^m$ be closed and $\varepsilon > 0$. Then there exists a family $\mathcal{J}$ of non-overlapping basic cubes such that the following conditions are satisfied:*

i) each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}$,

ii) $diam\, J \le \varepsilon \wedge [\varrho(A, J)]^2$ for each $J \in \mathcal{J}$.

Proof. Let $\mathcal{I}$ be a family of basic cubes such that $\bigcup \mathcal{I} = \mathbb{R}^m \setminus A$ and each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{I}$ [5, Lemma 2.1]. Write each cube $I \in \mathcal{I}$ as the union

$$I = \bigcup_{i=1}^{k_I} J_{I,i}$$

of non-overlapping basic cubes of diameter less than $\varepsilon \wedge [\varrho(A, I)]^2$ and define

$$\mathcal{J} = \{J_{I,i} : I \in \mathcal{I}, i \in \{1, \ldots, k_I\}\}.$$

Then the requirements of the lemma are obviously satisfied.

□

By *interval function* we will mean mapping from Γ into $\mathbb{R}$.

We say that intervals I, J are *contiguous*, if they do not overlap and $I \cup J$ is an interval. We say that an interval function F is *additive*, if $F(I \cup J) = F(I) + F(J)$ whenever I and J are contiguous intervals.

We say that a sequence of intervals $\{I_n : n \in \mathbb{N}\}$ is *o-convergent* to a point $x \in \mathbb{R}^m$, if

1. $x \in \bigcap_{n=1}^{\infty} I_n$,

2. $\lim_{n\to\infty} diam\, I_n = 0$,

3. $\limsup_{n\to\infty} \frac{(diam\, I_n)^m}{|I_n|} < \infty$.

We will write $I_n \overset{o}{\Rightarrow} x$. (Cf e.g. [5].)

Let F be an arbitrary interval function and $x \in \mathbb{R}^m$. We define

$$o\text{-}\limsup_{I\Rightarrow x} F(I) = \sup\left\{\limsup_{n\to\infty} F(I_n) :\ I_n \overset{o}{\Rightarrow} x\right\}.$$

In similar way we define $o\text{-}\liminf_{I\Rightarrow x} F(I)$ and $o\text{-}\lim_{I\Rightarrow x} F(I)$.

We say that function f is an *o-derivative*, if there exists an additive interval function F (called the *primitive* of f) such that for each $x \in \mathbb{R}^m$,

$$o\text{-}\lim_{I\Rightarrow x} \frac{F(I)}{|I|} = f(x).$$

Recall that o-derivatives are Baire one functions (cf [1, Lemma 2.1, p. 151] and [5, Lemma 3.1]).

We say that $x \in \mathbb{R}^m$ is an *o-Lebesgue point* of function f, if f is locally summable at x and

$$o\text{-}\lim_{I\Rightarrow x} \frac{\int_I |f - f(x)|}{|I|} = 0.$$

We say that f is an *o-Lebesgue function*, if each $x \in \mathbb{R}^m$ is an o-Lebesgue point of f.

We say that $x \in \mathbb{R}^m$ is an *o-dispersion point* of a set $A \subset \mathbb{R}^m$ iff

$$o\text{-}\lim_{I\Rightarrow x} \frac{|A \cap I|}{|I|} = 0.$$

We say that A is *d_o-open*, if each $x \in A$ is an o-dispersion point of $\mathbb{R}^m \setminus A$. The family of all d_o-open sets forms a topology on $\mathbb{R}^m$, so called *o-density topology* (cf [4]). The terms *"d_o-closed"*, *"d_o-interior"* (d_o-int) etc. will refer to this topology. We say that function f is *o-approximately continuous* if and only if it is continuous with respect to this topology. The family of all o-approximately continuous functions will be denoted by $\mathcal{C}_{o\text{-ap}}$. Recall that:

- for every measurable set $A \subset \mathbb{R}^m$, $|A \setminus d_o\text{-int}\, A| = 0$,
- each element of $\mathcal{C}_{o\text{-ap}}$ is a Baire one function,
- each bounded element of $\mathcal{C}_{o\text{-ap}}$ is an o-derivative.

The following lemma can be found both in [3] and in [2].

Lemma 3 *Let $B \subset \mathbb{R}^m$ be measurable, let $F_1, \ldots, F_n \subset d_o$-int B be closed and let $c_1, \ldots, c_n \in \mathbb{R}$. Then there exists an o-Lebesgue function φ such that*

- $\varphi(x) = c_i$, *if* $x \in F_i$, $i \in \{1, \ldots, n\}$,
- $\varphi(x) = 0$, *if* $x \notin B$,
- $\|\varphi\| \leq \max\{|c_i| : i \in \{1, \ldots, n\}\}$.

We say that function f is *o-non-degenerate* at a point $x \in \mathbb{R}^m$ if and only if x is not an o-dispersion point of the pre-image of the set $(f(x) - \varepsilon, f(x) + \varepsilon)$ by f for any $\varepsilon > 0$. We say that f is *o-non-degenerate*, if it is o-non-degenerate at each point $x \in \mathbb{R}^m$.

Lemma 4 *Assume that a sequence of pairwise disjoint sets $\{H_n : n \in \mathbb{N}\}$, a sequence of o-approximately continuous functions $\{h_n : n \in \mathbb{N}\}$ and $c \in (0, 1]$ satisfy the following conditions:*

i) $h_n(x) = 0$, *if* $x \notin H_n$, $n \in \mathbb{N}$,

ii) $|\{x \in H_n : h_n(x) = 0\}| \geq c \cdot |H_n|$, $n \in \mathbb{N}$,

iii) for every $x \notin \bigcup_{n=1}^{\infty} H_n$ and every $\tau > 0$, there exists a cube $I \ni x$ such that $\operatorname{diam} I < \tau$ and for each $n \in \mathbb{N}$, either $|H_n \cap I| = 0$ or $H_n \subset I$,

iv) for each $j \in \mathbb{N}$ and each $x \in H_j$, there is a $p > j$ such that for each $n > p$, $\operatorname{diam} H_n < [\varrho(x, H_n)]^2$.

Set $h = \sum_{n=1}^{\infty} h_n$. Then h is o-non-degenerate.

Proof. Take an $x \in \mathbb{R}^m$ and an $\varepsilon > 0$. Denote by A the pre-image of the set $(h(x) - \varepsilon, h(x) + \varepsilon)$ by h.

First assume that $x \notin \bigcup_{n=1}^{\infty} H_n$. For each $n \in \mathbb{N}$, let I_n be a cube chosen according to iii) with $\tau = 1/n$. Then clearly $I_n \overset{o}{\Rightarrow} x$ and by conditions i) and ii), we get

$$|A \cap I_n| \geq |\{t \in I_n : h(t) = 0\}| \geq \left| I_n \setminus \bigcup_{k=1}^{\infty} H_k \right| + \sum_{H_k \subset I_n} c \cdot |H_k| \geq c \cdot |I_n|.$$

Hence

$$o\text{-}\limsup_{I \Rightarrow x} \frac{|A \cap I|}{|I|} \geq c > 0.$$

Now let $x \in H_j$ for some $j \in \mathbb{N}$. Let p be a number chosen according to iv) and let $\tau > 0$. Since the function $g = \sum_{i=1}^{p} h_i$ is o-approximately continuous and $g(x) = h(x)$, there exists an $\eta > 0$ such that for each cube $I \ni x$, if $diam\, I < \eta$, then

$$|\{t \in I : |g(t) - h(x)| < \varepsilon\}| > (1 - \tau) \cdot |I|.$$

Let I be a cube such that $x \in I$ and $diam\, I < \eta$. Denote by B the union of those H_n with $n > p$ which intersection with the frame of I is non-void. Observe that

$$|B \cap I| \leq 2m \cdot \max\{diam\, H_n : n > p\} \cdot (diam\, I)^{m-1} \leq 2m \cdot (diam\, I)^{m+1},$$

so

$$\begin{aligned} |A \cap I| &\geq |\{t \in I : |g(t) - h(x)| < \varepsilon\} \cap \{t \in I : h(t) = g(t)\}| \\ &\geq (1 - \tau) \cdot |I| + \sum_{n>p} |\{t \in I \cap H_n : h_n(t) = 0\}| + \left| I \setminus \bigcup_{n>p} H_n \right| - |I| \\ &\geq -\tau \cdot |I| + c \cdot |I \setminus B| \geq (c - \tau - 2m^{1+m/2} \cdot diam\, I) \cdot |I|. \end{aligned}$$

Hence, since τ was arbitrary, we get

$$o\text{-}\limsup_{I \Rightarrow x} \frac{|A \cap I|}{|I|} \geq c.$$

□

Lemma 5 *The sum of an o-approximately continuous function with an o-non-degenerate function is o-non-degenerate.*

The proof is left to the reader.

□

Lemma 6 *Given a function v and a non-empty set $A \subset \mathbb{R}^m$, if $\omega(v,A) \leq M^2$ for some $M \in \mathbb{R}$, then*

a) $\omega\left(\sqrt{|v|}, A\right) \leq |M|$,

b) $\omega\left(|v| \vee \sqrt{|v|}, A\right) \leq M^2 \vee |M|$,

c) $\omega\left(1 \wedge \sqrt{|v|}, A\right) \leq |M|$,

Proof. Set $w_1 = |v| \vee \sqrt{|v|}$ and $w_2 = 1 \wedge \sqrt{|v|}$. Let $x, y \in A$.
a) If $\sqrt{|v(x)|} \leq |M|$ and $\sqrt{|v(y)|} \leq |M|$, then obviously

$$\left|\sqrt{|v(x)|} - \sqrt{|v(y)|}\right| \leq |M|.$$

In the opposite case we have

$$\left|\sqrt{|v(x)|} - \sqrt{|v(y)|}\right| = \left|\frac{|v(x)| - |v(y)|}{\sqrt{|v(x)|} + \sqrt{|v(y)|}}\right| \leq M^2/|M| = |M|.$$

b) If $|v(x)| \leq 1$ and $|v(y)| \leq 1$, then

$$|w_1(x) - w_1(y)| = \left|\sqrt{|v(x)|} - \sqrt{|v(y)|}\right| \leq |M|.$$

In the opposite case we have

$$|w_1(x) - w_1(y)| \leq \big|\,|v(x)| - |v(y)|\,\big| \leq M^2.$$

c) If $|v(x)| \geq 1$ and $|v(y)| \geq 1$, then

$$|w_2(x) - w_2(y)| = 0.$$

In the opposite case we have

$$|w_2(x) - w_2(y)| \leq \left|\sqrt{|v(x)|} - \sqrt{|v(y)|}\right| \leq |M|.$$

□

The following two lemmas are due to R. Carrese.

Lemma 7 *[2, Proposition 3] Let $H_1, H_2, \ldots$ be a sequence of pairwise disjoint compact subsets of $\mathbb{R}^m$ and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of non-negative numbers such that the function $\sum_{n=1}^{\infty} K_n \cdot \chi_{H_n}$ is a Baire one function. Then there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers satisfying the following conditions:*

a) for every sequence of functions $f_1, f_2, \ldots$, if for each $n \in \mathbb{N}$,

i) f_n is an o-derivative,

ii) $f_n(x) = 0$, if $x \notin H_n$,

iii) $\|f_n\| \leq K_n$,

iv) $\left|\int_I f_n\right| \leq \varepsilon_n$ for every interval I,

then function $f = \sum_{n=1}^{\infty} f_n$ is an o-derivative,

b) for every sequence of functions $w_1, w_2, \ldots$, if for $n \in \mathbb{N}$,

i) w_n is an o-Lebesgue function,

ii) $w_n(x) = 0$, if $x \notin H_n$,

iii) $\|w_n\| \leq K_n$,

iv) $\int_{K_n} |w_n| \leq \varepsilon_n$,

then function $w = \sum_{n=1}^{\infty} w_n$ is an o-Lebesgue function.

Lemma 8 *[2, Proposition 2] Let u be a Baire one function. There are a Baire one function v, a sequence $\{H_n : n \in \mathbb{N}\}$ of pairwise disjoint compact subsets of $\mathbb{R}^m$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of positive numbers such that the*

i) $u - v$ is an o-Lebesgue function,

ii) v is o-approximately continuous at all points of $\bigcup_{n\in\mathbb{N}} H_n$,

iii) $v(x) = 0$ whenever $x \in H_n$ for some $n \in \mathbb{N}$ and $x \notin d_o\text{-}int H_n$,

iv) $|v| \leq \sum_{n=1}^{\infty} c_n \cdot \chi_{H_n}$,

v) $\sum_{n\in\mathbb{N}} c_n \cdot \chi_{H_n}$ is a Baire one function,

vi) v is bounded provided that u is bounded.

The next lemma is a modified version of Proposition 4 of [2].

Lemma 9 *Assume that a set $A \subset \mathbb{R}^m$ is non-void, bounded and measurable, function v is o-approximately continuous, $v(x) = 0$ for $x \notin A$, $\|v\| \leq c < \infty$ and $\varepsilon > 0$. Then there exist o-approximately continuous functions f and g such that the following conditions are satisfied:*

i) $f(x) = g(x) = 0$ *for* $x \notin A$,

ii) $\|f\| \leq 2c \vee \sqrt{2c}$, $\|g\| \leq 1 \wedge \sqrt{2c}$,

iii) $\left|\int_I f\right| \leq \varepsilon$, $\left|\int_I g\right| \leq \varepsilon$ *for every interval* I,

iv) $\left|\int_I (v - f \cdot g)\right| \leq \varepsilon$ *for every interval* I,

v) $|\{x \in A : f(x) = 0\}| \geq |A|/4$, $|\{x \in A : g(x) = 0\}| \geq |A|/4$,

vi) $|\{x \in A : v(x) = f(x) \cdot g(x)\}| \geq |A|/4$.

Proof. Write A as the union $A = \bigcup_{n=1}^k A_n$ of measurable, pairwise disjoint, non-void sets of diameter less than

$$\frac{\varepsilon}{16m \cdot (1 \vee c) \cdot (1 \vee diam\, A)^{m-1}}.$$

For $n \in \{1, \ldots, k\}$, do the following.

If $|A_n| = 0$, then set $p_n = 1$, $B_n = C_n = D_n = A_{n,1} = P_{n,1} = Q_{n,1} = \emptyset$, $f_{n,1} = \varphi_{n,1} = 0$ and $v_n = v$. Otherwise find disjoint measurable sets $B_n, C_n \subset A_n$ and a closed set $D_n \subset d_o\text{-int}\, C_n$ such that

$$7|A_n|/24 > |C_n| \geq |D_n| \geq |B_n| > |A_n|/4.$$

Let φ_n be a non-negative o-approximately continuous function such that:

- $\varphi_n(x) = 1$ if $x \in D_n$,
- $\varphi_n(x) = 0$ if $x \notin C_n$,
- $\varphi_n \leq 1$ on $\mathbb{R}^m$

(cf Lemma 3). Put

$$v_n = v + \varphi_n \cdot \frac{\int_{B_n} v}{\int_{C_n} \varphi_n}.$$

Express the set $A_n \setminus B_n$ as the union $A_n \setminus B_n = \bigcup_{j=1}^{p_n} A_{n,j}$ of measurable, pairwise disjoint, non-void sets such that for $j \in \{1, \dots, p_n\}$, v_n does not change its sign on $A_{n,j}$ and

$$\omega(v_n, A_{n,j}) \le \frac{\varepsilon}{1 \vee 2|A|} \wedge \left(\frac{\varepsilon}{1 \vee 2|A|}\right)^2 .$$

For $j \in \{1, \dots, p_n\}$, find closed, disjoint sets $P_{n,j}, Q_{n,j} \subset d_o\text{-int}\, A_{n,j}$ such that $|P_{n,j}| = |Q_{n,j}|$ and

$$|A_{n,j} \setminus (P_{n,j} \cup Q_{n,j})| \le \frac{\varepsilon \wedge |A_n|}{8kp_n \cdot (1 \vee c)},$$

use Lemma 3 to find an o-approximately continuous function $\varphi_{n,j}$ such that

- $\varphi_{n,j}(x) = 1$ if $x \in P_{n,j}$,
- $\varphi_{n,j}(x) = 0$ if $x \notin A_{n,j}$,
- $\varphi_{n,j}(x) = -1$ if $x \in Q_{n,j}$,
- $|\varphi_{n,j}| \le 1$ on $\mathbb{R}^m$

and set

$$f_{n,j} = \begin{cases} \varphi_{n,j} \cdot \left(v_n \vee \sqrt{|v_n|}\right) & \text{if } v_n \ge 0 \text{ on } A_{n,j}, \\ -\varphi_{n,j} \cdot \left(|v_n| \vee \sqrt{|v_n|}\right) & \text{if } v_n \le 0 \text{ on } A_{n,j}. \end{cases}$$

Define

$$f = \sum_{n=1}^{k} \sum_{j=1}^{p_n} f_{n,j}$$

and

$$g = \sum_{n=1}^{k} \left(\left(1 \wedge \sqrt{|v_n|}\right) \cdot \sum_{j=1}^{p_n} \varphi_{n,j} \right) .$$

Then clearly f and g are o-approximately continuous and i) is fulfilled. Since for $n \in \{1, \dots, k\}$, if $|A_n| > 0$, then

$$\|v_n\| \le \|v\| + \frac{\int_{B_n} |v|}{\int_{C_n} \varphi_n} \le c + c \cdot \frac{|B_n|}{|D_n|} \le 2c,$$

so condition ii) holds.

Let I be an arbitrary interval. Denote by B the union of those $A_{n,j}$ which intersection with the frame of I is non-void. Let δ denote the $diam\, A$. Observe that

$$\begin{aligned} |B \cap I| &\le 2m \cdot \max\{diam\, A_{n,j} : n \in \{1, \ldots, k\}, j \in \{1, \ldots, p_n\}\} \cdot \delta^{m-1} \\ &\le 2m \cdot \frac{\varepsilon}{16m \cdot (1 \vee c) \cdot (1 \vee \delta)^{m-1}} \cdot \delta^{m-1} = \frac{\varepsilon}{8 \cdot (1 \vee c)}. \end{aligned}$$

Since for every $n \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, p_n\}$,

$$\begin{aligned} \left| \int_{A_{n,j}} f \right| &\le \left| \int_{A_{n,j} \setminus (P_{n,j} \cup Q_{n,j})} f \right| + \left| \int_{P_{n,j} \cup Q_{n,j}} f \right| \\ &\le \left(2c \vee \sqrt{2c}\right) \cdot \frac{\varepsilon}{8kp_n \cdot (1 \vee c)} + \omega\left(|v_n| \vee \sqrt{|v_n|}, A_{n,j}\right) \cdot |P_{n,j}| \\ &\le \frac{\varepsilon}{4kp_n} + \frac{\varepsilon \cdot |A_{n,j}|}{1 \vee 4|A|} \end{aligned}$$

(cf Lemma 6), so

$$\begin{aligned} \left| \int_I f \right| &= \left| \int_{A \cap I} f \right| \le \sum_{n=1}^{k} \sum_{j=1}^{p_n} \left| \int_{A_{n,j}} f \right| + \sum_{A_{n,j} \setminus I \neq \emptyset} \left| \int_{A_{n,j} \cap I} f \right| \\ &\le \sum_{n=1}^{k} \left(p_n \cdot \frac{\varepsilon}{4kp_n} + \sum_{j=1}^{p_n} \frac{\varepsilon \cdot |A_{n,j}|}{1 \vee 4|A|} \right) + \int_{B \cap I} |f| < \varepsilon. \end{aligned}$$

Similarly we can prove that $\left|\int_I g\right| < \varepsilon$.

For $n \in \{1, \ldots, k\}$, we have

$$\begin{aligned} &\left| \int_{A_n} (v - f \cdot g) \right| \\ &\quad\le \left| \int_{B_n} (v - f \cdot g) + \int_{A_n \setminus B_n} (v - v_n) \right| + \left| \int_{A_n \setminus B_n} (v_n - f \cdot g) \right| \\ &\quad\le \sum_{j=1}^{p_n} \left(\int_{A_{n,j} \setminus (P_{n,j} \cup Q_{n,j})} |v_n - f \cdot g| + \int_{P_{n,j} \cup Q_{n,j}} |v_n - f \cdot g| \right) \\ &\quad\le \sum_{j=1}^{p_n} \|v_n - f \cdot g\| \cdot |A_{n,j} \setminus (P_{n,j} \cup Q_{n,j})| \\ &\quad\le p_n \cdot 4c \cdot \frac{\varepsilon}{8kp_n \cdot (1 \vee c)} \le \frac{\varepsilon}{2k}, \end{aligned}$$

so

$$\begin{aligned}\left|\int_I (v - f\cdot g)\right| &= \left|\int_{A\cap I} (v - f\cdot g)\right| \\ &\le \sum_{n=1}^{k}\left|\int_{A_n}(v - f\cdot g)\right| + \sum_{A_n\setminus I\neq\emptyset}\left|\int_{A_n\cap I}(v - f\cdot g)\right| \\ &\le k\cdot\frac{\varepsilon}{2k} + \int_{B\cap I}|v - f\cdot g| \le \frac{\varepsilon}{2} + \|v - f\cdot g\|\cdot|B\cap I| \le \varepsilon.\end{aligned}$$

Note that $\{x\in A:\ f(x)=0\}\cap\{x\in A:\ g(x)=0\}\supset\bigcup_{n=1}^{k}B_n$ and $|B_n|\ge|A_n|/4$ for $n\in\{1,\ldots,k\}$, so v) holds. Finally observe that

$$\begin{aligned}&|\{x\in A:\ v(x)=f(x)\cdot g(x)\}| = \sum_{n=1}^{k}|\{x\in A_n:\ v(x)=f(x)\cdot g(x)\}| \\ &\ge \sum_{n=1}^{k}|\{x\in A_n:\ v(x)=v_n(x)\}\cap\{x\in A_n:\ v_n(x)=f(x)\cdot g(x)\}| \\ &\ge \sum_{n=1}^{k}\left|\bigcup_{j=1}^{p_n}(P_{n,j}\cup Q_{n,j})\setminus C_n\right| \ge \sum_{n=1}^{k}\left(|A_n\setminus B_n| - p_n\cdot\frac{|A_n|}{8kp_n} - |C_n|\right) \\ &\ge |A|/4.\end{aligned}$$

□

We will need a modified version of Lemma 8.

Lemma 10 *Whenever u is a Baire one function there exist a Baire one function v, a sequence of pairwise disjoint, compact sets $\{H_n:\ n\in\mathbb{N}\}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that the following conditions are satisfied:*

i) $u - v$ *is an o-Lebesgue function,*

ii) v *is o-approximately continuous at all points of* $\bigcup_{n=1}^{\infty}H_n$,

iii) $v(x)=0$, *if* $x\in H_n$ *for some* $n\in\mathbb{N}$ *and* $x\notin d_o\text{-int}H_n$,

iv) $|v|\le\sum_{n=1}^{\infty}c_n\cdot\chi_{H_n}$,

v) $\sum_{n=1}^{\infty}c_n\cdot\chi_{H_n}$ *is a Baire one function,* *vi)* *v is bounded provided that u is bounded,*

vii) *for every $x \notin \bigcup_{n=1}^{\infty} H_n$ and every $\tau > 0$, there exists a cube $I \ni x$ such that $diam\, I < \tau$ and for each $n \in \mathbb{N}$, either $H_n \cap I = \emptyset$ or $H_n \subset I$,*

viii) *for each $j \in \mathbb{N}$ and each $x \in H_j$, there exists a $p > j$ such that for each $n > p$, $diam\, H_n < [\varrho(x, H_n)]^2$.*

Proof. First use Lemma 8 to find a Baire one function v, a sequence of pairwise disjoint, compact sets $\{\overline{H}_n : n \in \mathbb{N}\}$ and a sequence $(\overline{c}_n)_{n\in\mathbb{N}}$ of non-negative real numbers satisfying conditions i)–vi). Analysing the proof of this lemma it is easy to observe that we may also require that $\bigcup_{n=1}^{\infty} \overline{H}_n \subset (\mathbb{R} \setminus \mathbb{Q})^m$. Set $\mathcal{J}_0 = \Gamma_1$ and $n_0 = 0$. For each $k \in \mathbb{N}$, set $A_k = \bigcup_{i=1}^{k-1} \overline{H}_i$ and apply Lemma 2 to find a family of non-overlapping basic cubes $\mathcal{J}_k = \{J_{k,n} : n \in \mathbb{N}\}$ such that $\bigcup \mathcal{J}_k = \mathbb{R}^m \setminus A_k$, every $x \in \mathbb{R}^m \setminus A_k$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}_k$ and

$$diam\, J_{k,n} \le \frac{1}{k} \wedge [\varrho(A_k, J_{k,n})]^2$$

for each $n \in \mathbb{N}$. We may also assume that $\mathcal{J}_k$ is a refinement of $\mathcal{J}_{k-1}$, i.e. each element of $\mathcal{J}_k$ is contained in one of elements of $\mathcal{J}_{k-1}$ (cf Remark on p. 600). By the compactness of $\overline{H}_k$, only finitely many sets of the family $\{J_{k,n} \cap \overline{H}_k : n \in \mathbb{N}\}$ are non-void. Denote those sets by $H_{n_{k-1}+1}, \ldots, H_{n_k}$ and set $c_i = \overline{c}_k$ for $i \in \{n_{k-1}+1, \ldots, n_k\}$.

It is easy to see that conditions i)–vi) are still fulfilled. To prove vii) take an $x \notin \bigcup_{n=1}^{\infty} H_n$ and $\tau \in (0, 1)$. Let $k > 1/\tau$ and let $n \in \mathbb{N}$ be such that $x \in J_{k,n}$. Set $I = J_{k,n}$. Then $I \cap \bigcup_{i=1}^{n_{k-1}} H_i = \emptyset$ and for $l > n_{k-1}$, either $H_l \cap I = \emptyset$ or $H_l \subset I$ (cf the construction of the family $\{H_n : n \in \mathbb{N}\}$).

Finally let $j \in \mathbb{N}$ and $x \in H_j$. Then $j < n_k$ for some $k \in \mathbb{N}$. Set $p = n_k$. It is obvious that p satisfies the requirements of condition viii).

□

Theorem 11 *Whenever $u : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a Baire one function there exist o-non-degenerate o-derivatives $f, g, h : \mathbb{R}^m \longrightarrow \mathbb{R}$ such that $u = f \cdot g + h$. Moreover one can find such a representation that g is bounded and in case u is bounded such that f and h are also bounded.*

Proof. Let function v, sequence of compact sets $\{H_n : n \in \mathbb{N}\}$ and sequence of non-negative real numbers $(c_n)_{n\in\mathbb{N}}$ be as in Lemma 10. Apply Lemma 7 with $K_n = c_n \vee \sqrt{c_n}$ $(n \in \mathbb{N})$ and find a sequence of positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ satisfying conditions of this lemma. For each $n \in \mathbb{N}$, use Lemma 9 with $A = H_n$ and $\varepsilon = \varepsilon_n$, getting in result o-approximately continuous functions

f_n and g_n fulfilling its requirements. Set $f = \sum_{n=1}^{\infty} f_n$, $g = \sum_{n=1}^{\infty} g_n$ and $h = u - f \cdot g$. By condition a) of Lemma 7, we get that f, g and $v - f \cdot g$ are $\mathcal{O}$-derivatives (conditions ii)–iii) of Lemma 10 and $\mathcal{O}$-approximate continuity of functions f_n and g_n imply $\mathcal{O}$-approximate continuity of $v_n = v\chi_{H_n} - f_n \cdot g_n$ for each $n \in \mathbb{N}$).

By consecutive use of Lemma 4, for the families $\{f_n : n \in \mathbb{N}\}$, $\{g_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$, we get that f, g and $v - f \cdot g$ are $\mathcal{O}$-non-degenerate (the assumptions of this lemma follow by conditions i), v) and vi) of Lemma 9 and conditions vii)–viii) of Lemma 10). Since $u - v$ is $\mathcal{O}$-approximately continuous and $v - f \cdot g$ is $\mathcal{O}$-non-degenerate, $h = (u - v) + (v - f \cdot g)$ is $\mathcal{O}$-non-degenerate, too (cf Lemma 5).

If u is bounded, we can choose function v also bounded. Then the families $\{f_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ have common bound, so f and h are bounded, which completes the proof.

□

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