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PRODUCTS OF DERIVATIVES OF INTERVAL FUNCTIONS WITH CONTINUOUS FUNCTIONS

Abstract

It is known that the family of all derivatives (from $\mathbb{R}$ into $\mathbb{R}$) whose product with every continuous function is a derivative is the same as the family of all locally summable derivatives such that

$$\limsup_{h\to 0+} \frac{\int_{x-h}^{x+h} |f|}{2h} < \infty$$

for each $x \in \mathbb{R}$. In this paper we prove an analogous theorem in multidimensional case.

In [4] J. Mařík proved the following theorem.

Theorem 1 *Denote by $\mathcal{F}$ the family of all derivatives (from $\mathbb{R}$ into $\mathbb{R}$) whose product with every continuous function is a derivative and by $\mathcal{F}_2$ the family of all locally summable derivatives such that*

$$\limsup_{h\to 0+} \frac{\int_{x-h}^{x+h} |f|}{2h} < \infty$$

for each $x \in \mathbb{R}$. Then $\mathcal{F} = \mathcal{F}_2$.

In this sequel I prove an analogous theorem in multidimensional case. In the proof I use the Mařík's method.

First we need some notation. The real line $(-\infty, +\infty)$ is denoted by $\mathbb{R}$ and the set of positive integers by $\mathbb{N}$. To the end of this sequel m is a fixed positive

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integer. The word function means mapping from $\mathbb{R}^m$ into $\mathbb{R}$ unless otherwise explicitly stated. The words measure, almost everywhere (a.e.), summable etc. refer to the Lebesgue measure and integral in $\mathbb{R}^m$. The Euclidean metric in $\mathbb{R}^m$ will be denoted by ϱ. For every set $A \subset \mathbb{R}^m$, let *diam* A be its diameter (i.e. $diam\, A = \sup\{\varrho(x,y) : x, y \in A\}$), *int* A its interior, *cl* A its closure, χ_A its characteristic function and $|A|$ its outer Lebesgue measure. Symbol $\int_A f$ will always mean the Lebesgue integral. We say that f is a Baire one function, if it is a pointwise limit of some sequence of continuous functions. By $\|f\|$ we denote the sup norm of a function f (i.e. $\|f\| = \sup\{|f(t)| : t \in \mathbb{R}^m\}$).

The word *interval* (*cube*) will always mean non-degenerate compact interval (cube) in $\mathbb{R}^m$, i.e. Cartesian product of m non-degenerate compact intervals (compact intervals of equal length) in $\mathbb{R}$. We denote by Γ the family of all intervals.

By *interval function* we will mean mapping from Γ into $\mathbb{R}$.

We say that intervals $I, J \in \Gamma$ are *contiguous*, if they do not overlap (i.e. $I \cap J \notin \Gamma$) and $I \cup J$ is an interval. We say that an interval function F is *additive*, if $F(I \cup J) = F(I) + F(J)$ whenever I and J are contiguous intervals.

We say that a sequence of intervals $\{I_n : n \in \mathbb{N}\}$ is

- *s-convergent* to a point $x \in \mathbb{R}^m$, if

  i) $x \in \bigcap_{n=1}^{\infty} I_n$,

  ii) $\lim_{n\to\infty} diam\, I_n = 0$.

- *o-convergent* to a point $x \in \mathbb{R}^m$, if the conditions i) and ii) above are fulfilled and moreover,

  iii) $\limsup_{n\to\infty} \frac{(diam\, I_n)^m}{|I_n|} < \infty$.

- *w-convergent* to a point $x \in \mathbb{R}^m$, if the conditions i) and ii) above are fulfilled and moreover,

  iv) I_n is a cube for each $n \in \mathbb{N}$.

We will write $I_n \overset{s}{\Rightarrow} x$, $I_n \overset{o}{\Rightarrow} x$ and $I_n \overset{w}{\Rightarrow} x$, respectively. (Cf e.g. [3].)

Let F be an arbitrary interval function and $x \in \mathbb{R}^m$. We define

$$s\text{-}\limsup_{I\Rightarrow x} F(I) = \sup\left\{\limsup_{n\to\infty} F(I_n) : I_n \overset{s}{\Rightarrow} x\right\}.$$

In similar way we define $o\text{-}\limsup_{I\Rightarrow x} F(I)$, $w\text{-}\limsup_{I\Rightarrow x} F(I)$, $s\text{-}\liminf_{I\Rightarrow x} F(I)$ etc.

We say that function f is an *s-derivative*, if there exists an additive interval function F (called the *primitive* of f) such that

$$s\text{-}\lim_{I \Rightarrow x} \frac{F(I)}{|I|} = f(x)$$

holds for each $x \in \mathbb{R}^m$. Analogously we define that function is an o-derivative or a w-derivative. The value of the primitive of a derivative f on interval I we will denote by $\mathcal{S}_s(f, I)$, $\mathcal{S}_o(f, I)$ and $\mathcal{S}_w(f, I)$, respectively (cf [3]). Recall that:

- w-derivatives (so also o-derivatives and s-derivatives) are Baire one functions (cf [1, Lemma 2.1, p. 151] and [3, Lemma 3.1]),
- If an o-derivative is summable on an interval I, then $\mathcal{S}_o(f, I) = \int_I f$ (cf [3, Proposition 5.3 and Corollary 6.2]). Similar result is true for s-derivatives and w-derivatives.

Lemma 2 *Given a function f of the first class of Baire which is not summable on an interval I we can find a family $\{I_n : n \in \mathbb{N}\}$ of non-overlapping cubes such that f is summable on each I_n ($n \in \mathbb{N}$) and $\sum_{n=1}^{\infty} \int_{I_n} |f| = \infty$.*

Proof. Let A be the set of all $x \in I$ at which $f \cdot \chi_I$ is locally summable. Then A is open so we can find a family $\{I_n : n \in \mathbb{N}\}$ of non-overlapping cubes such that $A = \bigcup_{n=1}^{\infty} I_n$ (cf [3, Lemma 2.1]). Suppose that $\sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$. Since f is not summable on I, the set $I \setminus A$ is nonvoid and so by Baire Theorem ([2, p. 301]), there is an $x \in I \setminus A$ such that $f|(I \setminus A)$ is continuous at x. Hence there is a bounded interval J such that $x \in int\, J$ and $f|(I \setminus A)$ is bounded on $J \cap I \setminus A$. But then

$$\int_J |f \cdot \chi_I| = \int_{J \cap I \setminus A} |f| + \int_{J \cap A} |f| \le \int_{J \cap I \setminus A} |f| + \sum_{n=1}^{\infty} \int_{I_n} |f| < \infty$$

—a contradiction, since $x \notin A$, i.e. $f \cdot \chi_I$ is not locally summable at x.

□

Lemma 3 *Whenever $|A \setminus int\, A| = 0$, function f is summable on A and $\varepsilon > 0$ there exists a continuous function g such that $||g|| \le 1$, $g(t) = 0$ for $t \notin A$ and*

$$\int_A (f \cdot g) > \int_A |f| - \varepsilon.$$

Proof. Since f is summable, there exists a $\gamma > 0$ such that $\int_C |f| < \varepsilon/2$ for each set $C \subset A$ of measure less than γ. Let $T_1 \subset \{t \in int\, A : f(t) \geq 0\}$ and $T_2 \subset \{t \in int\, A : f(t) < 0\}$ be closed sets such that $|A \setminus (T_1 \cup T_2)| < \gamma$. Let g be a continuous function which is equal to 1 on T_1, equal to -1 on T_2, equal to 0 out of $int\, A$ and such that $||g|| \leq 1$. Then

$$0 \leq \int_A |f| - \int_A (f \cdot g) = \int_A [f \cdot (sgn\, f - g)] \leq 2 \cdot \int_{A\setminus(T_1 \cup T_2)} |f| < \varepsilon.$$

□

Lemma 4 *Assume that I is an interval, $x \in I$ and h is a w-derivative which is locally summable at each $y \in I \setminus \{x\}$. Then for every descending sequence of cubes $I_n \overset{w}{\Rightarrow} x$, if $I_1 \subset I$, then function f is summable on $I_n \setminus I_{n+1}$ for each sufficiently large $n \in \mathbb{N}$ and moreover,*

$$\lim_{n\to\infty} \int_{I_n \setminus I_{n+1}} h = 0.$$

Proof. Let $\varepsilon > 0$. Then $x \notin cl\,(I_n \setminus I_{n+1})$ for sufficiently large $n \in \mathbb{N}$, whence h is for such n summable on $I_n \setminus I_{n+1}$. Using absolute continuity of Lebesgue integral find for each such n non-overlapping cubes $I_{n,1}, \ldots, I_{n,k_n} \subset I_n$ such that $I_{n+1} \subset I_{n,1}$, $I_n = \bigcup_{i=1}^{k_n} I_{n,i}$ and $\int_{I_{n,1}\setminus I_{n+1}} |f| < \varepsilon$. Let $\tau > 0$ be such that for each cube J, if $x \in J$ and $diam\, J < \tau$, then

$$\left| \frac{\mathcal{S}_w(h, J)}{|J|} - h(x) \right| < \varepsilon.$$

Hence for each sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} \left| \int_{I_n \setminus I_{n+1}} h \right| &< \varepsilon + \left| \int_{I_n \setminus I_{n,1}} h \right| = \varepsilon + |\mathcal{S}_w(h, I_n) - \mathcal{S}_w(h, I_{n,1})| \\ &< \varepsilon + (2\varepsilon + |h(x)|) \cdot |I_n|. \end{aligned}$$

□

Theorem 5 *For any function f, the following two conditions are equivalent:*

a) the product of f with each continuous function is a w-derivative,

b) f is a locally summable w-derivative and

$$w\text{-}\limsup_{I \Rightarrow x} \frac{\int_I |f|}{|I|} < \infty \tag{1}$$

for each $x \in \mathbb{R}^m$.

Proof.

a)⇒b) Note first that $f = f \cdot 1$ is a w-derivative, so f is a Baire one function. Suppose that f is not locally summable at some $x \in \mathbb{R}^m$. Then there exists an interval $I \ni x$ such that for each interval $J \subset I$, if $x \in J$, then f is not summable on J. We will define by induction a descending sequence of cubes $I_n \overset{w}{\Rightarrow} x$ and a sequence of continuous functions $\{g_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$, the following conditions hold:

1) $g_n(t) = 0$, if $t \notin I_n \setminus I_{n+1}$,

2) function $f \cdot g_n$ is summable and $\displaystyle\int_{I_n \setminus I_{n+1}} (f \cdot g_n) > 1$,

3) $\|g_n\| \leq 2^{1-n}$.

Set $I_1 = I$. Assume that we have already defined cubes $I_1, \dots, I_n$ and functions $g_1, \dots, g_{n-1}$ satisfying 1)–3). Use Lemma 2 to find non-overlapping cubes $I_{n,1}, \dots, I_{n,k_n} \subset I_n$ such that function f is summable on each $I_{n,i}$ ($i \in \{1, \dots, k_n\}$) and

$$\sum_{i=1}^{k_n} \int_{I_{n,i}} |f| > 2^n.$$

For $i \in \{1, \dots, k_n\}$, use Lemma 3 to find a continuous function $g_{n,i}$ such that $\|g_{n,i}\| \leq 1$, $g_{n,i}(t) = 0$ for $t \notin I_{n,i}$ and

$$\int_{I_{n,i}} (f \cdot g_{n,i}) \geq \int_{I_{n,i}} |f|/2.$$

Set

$$g_n = 2^{1-n} \cdot \sum_{i=1}^{k_n} g_{n,i}$$

and find a cube $I_{n+1} \subset I_n$ such that $x \in I_n$ and

$$diam\, I_n \leq \min\left\{ \varrho\left(x, \bigcup_{i=1}^{k_n} I_{n,i}\right), 2^{-n} \right\}.$$

Then

$$\int_{I_n \setminus I_{n+1}} (f \cdot g_n) = 2^{1-n} \cdot \sum_{i=1}^{k_n} \int_{I_{n,i}} (f \cdot g_{n,i}) \geq 2^{-n} \cdot \sum_{i=1}^{k_n} \int_{I_{n,i}} |f| > 1.$$

Obviously 1) and 3) are also fulfilled.

Set $g = \sum_{n=1}^{\infty} g_n$. Function g is continuous but $\int_{I_n \setminus I_{n+1}} (f \cdot g) > 1$ for each $n \in \mathbb{N}$, so according to Lemma 4, function $f \cdot g$ is not a w-derivative, a contradiction.

Suppose now that f is locally summable and

$$w\text{-}\limsup_{I \Rightarrow x} \frac{\int_I |f|}{|I|} = \infty$$

for some $x \in \mathbb{R}^m$. Then there exists a sequence of cubes $I_n \overset{w}{\Rightarrow} x$ such that for each $n \in \mathbb{N}$,

$$\int_{I_n} |f| > (n^2 + 1) \cdot |I_n| \quad \text{and} \quad \int_{\bigcup_{k>n} I_k} |f| < |I_n|$$

For each $n \in \mathbb{N}$, use Lemma 3 with $A_n = I_n \setminus \bigcup_{k>n} I_k$ to find a continuous function g_n such that $\|g_n\| \le 1$, $g_n(t) = 0$ for $t \notin A_n$ and

$$\int_{A_n} (f \cdot g_n) > n^2 \cdot |I_n|$$

(note that, since the set $\bigcup_{k>n} I_k$ is closed, $|A_n \setminus int\, A_n| = 0$). Set

$$g = \sum_{n=1}^{\infty} \frac{g_n}{n}.$$

Then g is continuous and since for each $n \in \mathbb{N}$,

$$\left| \int_{I_n \setminus A_n} (f \cdot g) \right| \le \int_{\bigcup_{k>n} I_k} |f| < |I_n|,$$

so

$$\frac{S_w(fg, I_n)}{|I_n|} = \frac{\int_{I_n} (f \cdot g)}{|I_n|} = \frac{n \cdot \int_{I_n \setminus A_n} (f \cdot g) + \int_{A_n} (f \cdot g_n)}{n \cdot |I_n|} > n - 1 \overset{n \to \infty}{\longrightarrow} \infty$$

—a contradiction.

b)⇒a) Let g be an arbitrary continuous function. Since f is locally summable, so is $f \cdot g$. Then for each $x \in \mathbb{R}^m$,

$$\begin{aligned} & w\text{-}\limsup_{I \Rightarrow x} \left| \frac{\int_I (f \cdot g)}{|I|} - f(x) \cdot g(x) \right| \\ = \; & w\text{-}\limsup_{I \Rightarrow x} \left| g(x) \cdot \left(\frac{\int_I f}{|I|} - f(x) \right) + \frac{\int_I [f \cdot (g - g(x))]}{|I|} \right| \\ \le \; & |g(x)| \cdot w\text{-}\lim_{I \Rightarrow x} \left| \frac{\int_I f}{|I|} - f(x) \right| + w\text{-}\limsup_{I \Rightarrow x} \frac{\int_I |f|}{|I|} \cdot w\text{-}\lim_{I \Rightarrow x} \|g \cdot \chi_I - g(x)\| \\ = \; & 0. \end{aligned}$$

□

Remark. It is easy to see that theorems analogous to Theorem 5, concerning o-derivatives and s-derivatives, can be proved in a similar way. However, since the o-convergence cannot be written in a Cauchy-like manner, the analogue of (1) is in this case a little more complicated. Example 8 shows that condition (3) cannot be replaced with the following:

$$o\text{-}\limsup_{I \Rightarrow x} \frac{\int_I |f|}{|I|} < \infty. \tag{2}$$

Theorem 6 *For any function f, the following two conditions are equivalent:*

a) the product of f with each continuous function is an o-derivative,

b) f is a locally summable o-derivative and

$$\limsup_{n \to \infty} \frac{\int_{I_n} |f|}{|I_n|} < \infty \tag{3}$$

for each $x \in \mathbb{R}^m$ and each sequence of intervals $I_n \overset{o}{\Rightarrow} x$.

Theorem 7 *For any function f, the following two conditions are equivalent:*

a) the product of f with each continuous function is an s-derivative,

b) f is a locally summable s-derivative and

$$s\text{-}\limsup_{I \Rightarrow x} \frac{\int_I |f|}{|I|} < \infty \tag{4}$$

for each $x \in \mathbb{R}^m$.

Example 8 *Assume that $m > 1$. Then there exists a locally summable o-derivative f and $x \in \mathbb{R}^m$ such that (3) holds and (2) does not.*

For each $n \in \mathbb{N}$, set

$$J_n = [2^{1-2n}, 2^{2-2n}] \times [2^{-n}, 2^{1-n}] \times \ldots \times [2^{-n}, 2^{1-n}]$$

and find a continuous function f_n such that $f_n(y) = 0$ for $y \notin J_n$,

$$\int_{J_n} |f_n| = 2^{n-1} \cdot |J_n| = 2^{-mn}$$

and for every interval $I \in \Gamma$,

$$\left|\int_I f_n\right| \le 2^{-2mn}.$$

Set $f = \sum_{n=1}^{\infty} f_n$ and $x = (0, \ldots, 0)$.

Let $I_n \overset{o}{\Rightarrow} x$. There exists an $\alpha \in \mathbb{R}$ such that

$$\frac{(diam\, I_n)^m}{|I_n|} < \alpha < \infty$$

for each $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. Let $p = \min\{k \in \mathbb{N} : I_n \cap J_k \neq \emptyset\}$. Then

$$\frac{1}{|I_n|} \cdot \int_{I_n} |f| \le \alpha \cdot \frac{1}{[\varrho(x, J_p)]^m} \cdot \sum_{k=p}^{\infty} \int_{J_k} |f_k| < \frac{\alpha}{1 - 2^{-m}},$$

so (3) holds. Meanwhile

$$\frac{1}{|I_n|} \cdot \left|\int_{I_n} f\right| \le \alpha \cdot \frac{1}{[\varrho(x, J_p)]^m} \cdot \sum_{k=p}^{\infty} 2^{-2mk} < \frac{\alpha \cdot 2^{-mp}}{1 - 2^{-2m}},$$

so f is an o-derivative.

Let $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, set

$$I_n = [0, 2^{1-n-k}] \times [0, 2^{1-n}] \times \ldots \times [0, 2^{1-n}]$$

and observe that

$$\frac{(diam\, I_n)^m}{|I_n|} < 2^k \cdot m^{m/2} < \infty$$

(so $I_n \overset{o}{\Rightarrow} x$) but for $n > k$,

$$\frac{1}{|I_n|} \cdot \int_{I_n} |f| > \frac{1}{2^{m+n-k-1} \cdot |J_n|} \int_{J_n} |f_n| = 2^{k-m},$$

so, since $k \in \mathbb{N}$ is arbitrary, (2) does not hold.

□

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