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ON A THEOREM OF MENKYNA*

Abstract

We charaterize the set where an almost everywhere continuous Baire 1 function is not a.e. continuous in the O'Malley sense.

In the paper [3] Menkyna gives a characterization of the set of points at which a Baire 1 function $f : (a,b) \longrightarrow \mathbb{R}$ is approximately continuous. In this article I show an analogous characterization of the set where an almost everywhere continuous Baire 1 function is a.e. continuous in O'Malley's sense (cf [4]). Since the set of all points at which f is not approximately continuous is of (Lebesgue) measure zero and the set where f is not a.e. continuous may be of positive measure, such a characterization of the set where f is a.e. continuous is not possible for all Baire 1 functions.

Let $\mathbb{R}$ denote the set of reals and let m be the Lebesgue measure in $\mathbb{R}$. If $A \subset \mathbb{R}$ is a measurable (in the Lebesgue sense) set and if $x \in \mathbb{R}$ then the number

$$d_u(A,x) = \limsup_{h\to 0} m(A \cap [x-h, x+h])/2h$$

is called the upper density of A at x. The lower density $d_l(A,x)$ is defined analogously. If $d_u(A,x) = d_l(A,x)$, we call this number the density of A at x and denote it by $d(A,x)$. The family T_d of all measurable sets $A \subset \mathbb{R}$ such that if $x \in A$ then $d(A,x) = 1$ is a topology said the density topology (cf [1]). The family T_{ae} of all sets $A \in T_d$ such that $m(A - intA) = 0$ ($intA$ denotes the euclidean interior of A) is a topology said the a.e. topology (O'Malley [4]). Let $f : (a,b) \longrightarrow \mathbb{R}$ be a function. The function f is said to be a.e. continuous at a point $x \in (a,b)$ if for every $\varepsilon > 0$ there is a set $B \in T_{ae}$ such that $x \in B$ and $f(B) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$ (cf [4]). Denote by $C_{ae}(f)$ the set of all points $x \in (a,b)$ at which f is a.e. continuous. Let $I_1, \dots, I_n, \dots$ be a

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sequence of all open intervals with rational endpoints. For $n, k, l = 1, 2, \ldots$ let A_{nkl} be the set of all points $x \in (a, b)$ for which there exists an open interval $J_n(x)$ containing x and such that

$$(1) \qquad m(cl(\{t \in J_n(x); f(t) \in \mathbb{R} - I_k\})) > m(J_n(x))/l$$

($cl(X)$ denotes the closure of X) , $m(J_n(x) < 1/n$, and if $I_k = (c_k, d_k)$ then $f(x) \in [c_k + m(I_k)/4, d_k - m(I_k)/4]$.

It is easy to verify that:

Remark 1 *The equality*

$$(a, b) - C_{ae}(f) = \bigcup_{k,l=1}^{\infty} \bigcap_{n=1}^{\infty} A_{nkl}$$

holds.

Remark 2 *If f is a Baire 1 function then every set $A_{nkl}, n, k, l = 1, 2, \ldots,$ is an G_δ set. Consequently, the set $(a, b) - C_{ae}(f)$ is an $G_{\delta\sigma}$ set.*

Remark 3 *If f is an almost everywhere continuous function then $m((a, b) - C_{ae}(f)) = 0$.*

Now, let Φ be a family of sets. Define

$$d_u^*(\Phi, x) = d_u(\bigcup\{A \in \Phi; d(A, x) = 0\}, x)$$

(cf [3]) .

The main result of this article is the following:

Theorem 1 *If $f : (a, b) \longrightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, is an almost everywhere continuous Baire 1 function then there is a sequence of open sets $V_n, n = 1, 2, \ldots,$ such that*

$$(2) \qquad m(cl(V_n) - V_n) = 0, n = 1, 2, \ldots,$$

and

$$(3) \qquad (a, b) - C_{ae}(f) = \bigcup_{n=1}^{\infty} \{x; d_u^*(\{T_n^s\}, x) > 0,$$

where T_n^s are the components of V_n, and conversely, for every sequence of open sets $V_n \subset (a, b), n = 1, 2, \ldots,$ satisfying (2) there is a Baire 1 function f such that (3) holds.

PROOF. If (V_n) is a sequence of open sets satisfying (2), then the same as in the proof of Theorem 5 from [3] we define for $n = 1, 2, \ldots$,

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (a,b) - \bigcup_s (a_n^s, b_n^s) \\ \sin(2^{s+1}\pi(x - a_n^s)/(b_n^s - a_n^s)) & \text{if } x \in (a_n^s, b_n^s), \end{cases}$$

where (a_n^s, b_n^s) is the middle open third of T_n^s.

It is easy to compute that every f_n is a derivative (compare [1]) and that

$$(a,b) - C_{ae}(f_n) = \{x; d_u^*(\{T_n^s\}_s, x) > 0\}.$$

The function $f = 4^{-1}f_1 + \cdots + 4^{-n}f_n + \cdots$ is a derivative (cf [1] ,p.17) and therefore a Baire 1 function. Moreover,

$$(a,b) - C_{ae}(f) = \bigcup_n ((a,b) - C_{ae}(f_n)) = \bigcup_n \{x; d_u^*(\{T_n^s\}_s, x) > 0\}.$$

For the proof of the converse implication we introduce some notation and prove several lemmas.

Lemma 1 *Let $f : (a,b) \longrightarrow \mathbb{R}$ be an almost everywhere continuous Baire 1 function. There is a sequence of almost everywhere continuous Baire 1 functions f_k such that every set $f_k((a,b))$ is isolated, and*

$$|f_k - f| < \min((4k)^{-1}, m(I_k)/8), \ k = 1, 2, \ldots$$

PROOF OF LEMMA 1. Denote by $C(f)$ the set of all continuity points of f. Since $m((a,b) - C(f)) = 0$, by Vitali's Theorem there is a countable disjoint collection $J_1, \ldots, J_n, \ldots$ of open intervals such that $m((a,b) - \bigcup_{n=1}^\infty J_n) = 0$ and $osc_{J_n} f < \min((4k)^{-1}, m(I_k)/8)/2, n = 1, 2, \ldots$. Then the set $F = (a,b) - \bigcup_n J_n$ is closed in (a,b) and $m(F) = 0$. There is a Baire 1 function $h_k : F \longrightarrow \mathbb{R}$ such that the set $h_k(F)$ is isolated and $|h_k - f| < \min((4k)^{-1}, m(I_k)/8)$ (cf [2], p.294). Then the function

$$f_k(x) = \begin{cases} y_n & \text{if } x \in J_n, \ n = 1, 2, \ldots \\ h_k(x) & \text{if } x \in F, \end{cases}$$

where $|y_n - f(x_n| < \min((4k)^{-1}, m(I_k)/8)/2$ for some $x_n \in J_n, n = 1, 2, \ldots$ and the set $h_k(F) \cup \{y_n; n = 1, 2, \ldots\}$ is isolated, satisfies all required conditions. This finishes the proof of Lemma 1.

Now, let $K_i^k, i, k = 1, 2, \ldots$, be closed sets such that $\bigcup_i K_i^k = (a,b)$ for $k = 1, 2, \ldots$, and the restrictions of the functions f_k from Lemma 1 to K_i^k are constant functions. Let

$$A_{kl}^i = K_i^k \cap \bigcap_n A_{nkl}.$$

Since f is an almost everywhere continuous Baire 1 function, every set A^i_{kl} is of type G_δ and measure zero.

Lemma 2 *The inclusions*

$$f(cl(A^i_{kl})) \subset I_k, k, l, i = 1, 2, \ldots$$

hold.

PROOF. Every function f_k is constant on the set $cl(A^i_{kl}) \subset K^k_i$ and $|f(x) - f_k(x)| < \min((4k)^{-1}, m(I_k)/8)$ for every $x \in (a, b)$. Fix $y \in A^i_{kl}$. Then

$$\begin{aligned} |f(x)| &\leq |f(x) - f_k(x)| + |f_k(x)| < m(I_k)/8 + |f_k(y)| \\ &\leq |f_k(y) - f(y)| + |f(y)| + m(I_k)/8 < m(I_k)/8 + m(I_k)/8 + |f(y)| \end{aligned}$$

for every $x \in cl(A^i_{kl})$. Since

$$f(y) \in [c_k + m(I_k)/4, d_k - m(I_k)/4],$$

$f(x) \in I_k = (c_k, d_k)$ for each $x \in cl(A^i_{kl})$.

Lemma 3 *Let $U \supset A^i_{kl}$ be an open set. Then there is an open set U' such that $U \supset U' \supset A^i_{kl}$ and for each component T_s of U' we have*

$$m(T_s \cap [(a, b) - cl(A^i_{kl})] \geq m(T_s \cap cl(\{x; f(x) \in \mathbb{R} - I_k\}) \geq m(T_s)/2l.$$

PROOF. From the definition of the set A^i_{kl} it is evident that for every $x \in A^i_{kl}$ we may choose an open interval $J(x) \subset U$ such that

$$m(cl(\{t \in J(x); f(t) \in \mathbb{R} - I_k\})) > m(J(x))/l.$$

Let $U' = \bigcup\{J(x); x \in A^i_{kl}\}$. If T_s is a component of the set U', then according to Lemma 2 from [3] we have

$$m(cl(\{t \in T_s; f(t) \in \mathbb{R} - I_k\})) > m(T_s)/2l.$$

So, we have the second inequality. Since f is almost everywhere continuous, we have also

$$m(T_s \cap \{x; f(x) \in \mathbb{R} - I_k\}) = m(T_s \cap cl(\{x; f(x) \in \mathbb{R} - I_k\})).$$

From this, by Lemma 2, we obtain the first inequality.

Lemma 4 *For every set A^i_{kl} there is an open set V such that*

$$(a, b) - C_{ae}(f) \supset \{x; d^*_u(\{T_s\}_s, x) > 0\} \supset A^i_{kl},$$

where T_s are the components of the set V.

PROOF. The proof of this Lemma is completely analogous as the proof of Theorem 4 and Corollary 1 in [3]. In the construction of the intervals J_n^{sr} we apply Lemma 3.

Remark 4 *Observe that the set V from Lemma 4 is such that* $m(cl(V)-V) = 0$.

Indeed, from the definition of A_{kl}^i it follows that $cl(A_{kl}^i) \subset cl(\{x \in (a,b); f(x) \in \mathbb{R} - I_k\})$. By Lemma 2, $f(cl(A_{kl}^i)) \subset I_k$. Since f is almost everywhere continuous, we have $m(cl(A_{kl}^i)) = 0$. From the construction of V (cf [3], pp.416 - 417) it follows that $m(cl(V) - V - cl(A_{kl}^i)) = 0$. so, $m(cl(V) - V) = 0$.

Now the proof of the converse implication of Theorem 1 is the same as that from [3]. It suffices to observe that $(a,b) - C_{ae}(f) = \bigcup_{k,l,i=1}^{\infty} A_{kl}^i$ and apply Lemma 4.

In the same way as in [3] we obtain:

Remark 5 *Theorem 1 is true, if we replace the concept "an almost everywhere continuous Baire 1 function" by "an almost everywhere continuous derivative".*

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