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QUALITATIVE SYMMETRIC DIFFERENTIATION

Abstract

It is shown that the set of points at which a real valued function of a real variable is qualitatively continuous and finitely qualitatively symmetrically differentiable but not qualitatively differentiable is a σ–symmetrically porous set.

1. Introduction

It follows from a result of M. Evans and L. Larson (Theorem 4.1 in [4]) that if $f : \mathbb{R} \rightarrow \mathbb{R}$ has the Baire property, then the set of points at which f is qualitatively continuous and (finitely) qualitatively symmetrically differentiable, but not qualitatively differentiable is σ–porous. Here, we shall prove a strengthening of this result by observing that the exceptional set is σ–symmetrically porous even if the assumption that f has the Baire property is dropped. (The existence of σ–porous sets which are not σ–symmetrically porous has been established in [3] and [8].) We shall obtain a qualitative version of a result of L. Zajíček [9] who showed that for an arbitrary $f : \mathbb{R} \rightarrow \mathbb{R}$ the set of points at which f is continuous and (finitely) symmetrically differentiable but not differentiable is σ–$(1-\epsilon)$–symmetrically porous for every $0 < \epsilon < 1$. (Zajíček did not state his result in this strong of a form, but his proof, indeed, verifies this strengthened statement. See [1].)

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2. Definitions and Notation

Here we establish the notation that will be utilized in this paper. The concepts of qualitative limits, qualitative continuity, and qualitative derivatives were introduced by S. Marcus [5, 6, 7]. Let $f : \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \mathbb{R}$. The *qualitative limit superior from the right of f at x_0* is defined as

$$q\text{-}\limsup_{x \to x_0+} f(x) = \inf\{y : \{x : f(x) > y\} \text{ is first category in a right neighborhood of } x_0\}.$$

The *qualitative limit inferior from the right of f at x_0* is defined as

$$q\text{-}\liminf_{x \to x_0+} f(x) = \sup\{y : \{x : f(x) < y\} \text{ is first category in a right neighborhood of } x_0\}.$$

If $q\text{-}\limsup_{x\to x_0+} f(x) = q\text{-}\liminf_{x \to x_0+} f(x)$, the common value is called the *qualitative limit from the right of f at x_0* and is denoted by $q\text{-}\lim_{x\to x_0+} f(x)$. Qualitative limits from the left, and qualitative limits are then defined and denoted in the obvious fashion. In the event that $f(x_0) = q\text{-}\lim_{x \to x_0} f(x)$ we say that *f is qualitatively continuous at x_0*, and let

$$C_q(f) = \{x : f \text{ is qualitatively continuous at } x\}.$$

The upper right qualitative derivate of f at x_0 is defined by

$$Q^+ f(x_0) = q\text{-}\limsup_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h}.$$

Then $Q^- f(x_0)$, $Q_+ f(x_0)$, and $Q_- f(x_0)$ are all defined in the obvious manner. In the event that all four of these derivates are equal, we call their common value the *qualitative derivative of f at x_0* and denote it by $f'_q(x_0)$; that is,

$$f'_q(x_0) = q\text{-}\lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h}.$$

Recall that the *upper symmetric derivate of f at x_0* is

$$\overline{D}^s f(x_0) = \limsup_{h \to 0+} \frac{f(x_0+h) - f(x_0-h)}{2h},$$

while the *lower symmetric derivate of f at x_0* is

$$\underline{D}^s f(x) = \liminf_{h \to 0+} \frac{f(x_0+h) - f(x_0-h)}{2h}.$$

If $\overline{D}^s f(x_0) = \underline{D}^s f(x_0)$, then the common value is called the *symmetric derivative of f at x_0* and is denoted $f'_s(x_0)$. Analogously, in the qualitative setting we say that the *upper qualitative symmetric derivate of f at x_0* is

$$\overline{Q}^s f(x_0) = q\text{-}\limsup_{h \to 0+} \frac{f(x_0+h) - f(x_0-h)}{2h},$$

while the *lower qualitative symmetric derivate of f at x_0* is

$$\underline{Q}^s f(x) = q\text{-}\liminf_{h \to 0+} \frac{f(x_0+h) - f(x_0-h)}{2h}.$$

If $\overline{Q}^s f(x_0) = \underline{Q}^s f(x_0)$, then the common value is called the *qualitative symmetric derivative of f at x_0* and is denoted $f'_{qs}(x_0)$.

If A is a subset of the real line $\mathbb{R}$ and $x \in \mathbb{R}$, then the *porosity of A at x* is defined to be

$$p(A,x) = \limsup_{r \to 0+} \frac{\lambda(A,x,r)}{r},$$

where $\lambda(A,x,r)$ is the length of the longest open interval contained in either $(x, x+r) \cap A^c$ or $(x-r, x) \cap A^c$ and A^c denotes the complement of A. A set is said to be *porous at x* if it has positive porosity at x and is called a *porous set* if it is porous at each of its points. Further, a set is called *σ-porous* if it is a countable union of porous sets. The *symmetric porosity of A at x* is defined as

$$p^s(A,x) = \limsup_{r \to 0+} \frac{\gamma(A,x,r)}{r},$$

where $\gamma(A,x,r)$ is the supremum of all positive numbers h such that there is a positive number t with $t+h \leq r$ such that both of the intervals $(x-t-h, x-t)$ and $(x+t, x+t+h)$ lie in A^c. A set A is *symmetrically porous* if it has positive symmetric porosity at each of its points. For a number $0 < \alpha \leq 1$ the set A is called *α-symmetrically porous* if it has symmetric porosity at least α at each of its points. The set A is called *σ-symmetrically porous* if it is a countable union of symmetrically porous sets, and is called *σ-α-symmetrically porous* if it is a countable union of α-symmetrically porous sets.

Finally, we shall say that a set A is *residual* in an interval I provided that the set difference $I \setminus A$ is a first category set.

3. Results

The following lemma is the key to our main result and its proof will closely follow that of Zajíček [9], except that where he reflected points, we shall reflect intervals and keep track of the function values within a second category subset of the reflected intervals.

Lemma 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and let $0 < \epsilon < 1$. The set*

$$S_\epsilon(f) \equiv \left\{ x \in C_q(f) : |f'_{qs}(x)| < \frac{\epsilon Q^+ f(x)}{16} \right\}$$

is σ-$(1-\epsilon)$-symmetrically porous.

PROOF. For each positive rational number B let

$$S_{\epsilon,B}(f) \equiv \left\{x \in C_q(f) : |f'_{qs}(x)| < \frac{\epsilon B}{16} < Q^+f(x)\frac{\epsilon}{16}\right\}$$

Then $S_\epsilon(f) = \bigcup_{B>0} S_{\epsilon,B}(f)$ and it will be enough to show that each $S_{\epsilon,B}(f)$ is σ–$(1-\epsilon)$–symmetrically porous.

For each $x \in S_{\epsilon,B}(f)$ let

$$T(x,\epsilon,B) = \left\{y : \left|\frac{f(2x-y)-f(y)}{2(y-x)}\right| < \frac{B\epsilon}{16}\right\},$$

and for each natural number m set

$$S_{\epsilon,B,m}(f) = \left\{x \in S_{\epsilon,B} : T(x,\epsilon,B) \text{ is residual in } \left(x-\frac{1}{m}, x+\frac{1}{m}\right)\right\}.$$

Then since $S_{\epsilon,B} \subseteq \bigcup_m S_{\epsilon,B,m}(f)$ it suffices to show each $S_{\epsilon,B,m}(f)$ is $(1-\epsilon)$ symmetrically porous. Suppose for a fixed m, $S_{\epsilon,B,m}(f)$ is not $(1-\epsilon)$ symmetrically porous. So there is an $x_0 \in S_{\epsilon,B,m}(f)$ such that $p^s(S_{\epsilon,B,m}(f), x_0) < 1-\epsilon$. Without loss of generality we may assume $x_0 = 0 = f(x_0)$. Then there is a $0 < \delta < 1/4m$ such that for all $0 < t < \delta$

$$(1) \qquad [(-t+\epsilon t/2, -\epsilon t/2) \cup (\epsilon t/2, t-\epsilon t/2)] \cap S_{\epsilon,B,m}(f) \neq \emptyset.$$

Since $Q^+f(0) > B$, there is a number $a_1 \in (0,\delta)$ such that $f(a_1) > Ba_1$, and such that the set $\{x : f(x) > Ba_1\}$ intersects every open neighborhood of a_1 in a second category set. Since $f(0) = 0$ and f is qualitatively continuous at 0 there is a number $\delta^* > 0$ such that

$$(2) \qquad \{x \in (-\delta^*, \delta^*) : |f(x)| < Ba_1/2\} \text{ is residual in } (-\delta^*, \delta^*).$$

Let $I_1 \equiv (a_1 - \delta^*, a_1 + \delta^*)$ and $B_1 = \{x \in I_1 : f(x) > Ba_1\}$. Because of (2), $0 \notin I_1$; that is, $\delta^* \leq a_1$, and, consequently, $a_1 + \delta^* < 1/2m$..

Having defined a_1, we shall now inductively define a sequence $\{a_k\}$ of points such that for each natural number k we have

$$(3) \qquad |a_{k+1}| \leq (1-\epsilon)|a_k|.$$

If the points $a_1, \ldots, a_k$ have been selected we proceed as follows. First, if $a_k = 0$, we put $a_{k+1} = 0$. If $a_k \neq 0$ then from (1) it follows that there is a point

$$(4)\ p_k \in [(-|a_k| + \epsilon|a_k|/2, -\epsilon|a_k|/2) \cup (\epsilon|a_k|/2, |a_k| - \epsilon|a_k|/2)] \cap S_{\epsilon,B,m}(f) \quad ,$$

where for definiteness we shall choose p_k to have the same sign as a_k if we have a choice. We then set

$$(5) \qquad a_{k+1} = \begin{cases} 2p_k - a_k & \text{if } p_k a_k > 0 \\ 2p_k + a_k & \text{if } p_k a_k < 0 \end{cases} .$$

In other words, a_{k+1} is the reflection of a_k about p_k if a_k and p_k are on the same side of the origin; otherwise, a_{k+1} is the reflection of $-a_k$ about p_k. From (4) and (5) inequality (3) clearly follows for each natural number k. For each k let $I_k = (a_k - \delta^*, a_k + \delta^*)$. Because of (3) there is a smallest natural number N such that $0 \in I_{N+1}$.

Now, we define the set T to be

$$T = T(0,\epsilon,B) \bigcap \left\{ \bigcap_{k=1}^{N} [T(p_k,\epsilon,B) \cap -T(p_k,\epsilon,B)] \right\},$$

where $-T(p_k,\epsilon,B)$ is the reflection of $T(p_k,\epsilon,B)$ about 0. Note that T is residual in the interval $(-a_1 - \delta^*, a_1 + \delta^*)$. We set

$$C_1 = B_1 \cap T.$$

For each $k = 1,\ldots,N$ define a function $r_k : \mathbb{R} \to \mathbb{R}$ by

$$(6) \qquad r_k(x) = \begin{cases} 2p_k - x & \text{if } p_k x > 0 \\ 2p_k + x & \text{if } p_k x \le 0 \end{cases} .$$

Consequently, $r_k(x)$ is the reflection of x about p_k if x and p_k are on the same side of the origin; otherwise, $r_k(x)$ is the reflection of $-x$ about p_k. In this notation,

$$a_{k+1} = r_k(a_k) \text{ and } I_{k+1} = r_k(I_k) \text{ for } k = 1,\ldots N,$$

the latter resulting from the fact that for $k = 1,\ldots,N$, we must have $\delta^* \le |a_k|$, and hence for all $x \in I_k$, $p_k x > 0$ if and only if $p_k a_k > 0$. For each $k = 1,\ldots,N$ we define

$$C_{k+1} = r_k(C_k)$$

and observe that the set C_k lies in the interval I_k and is second category in every open neighborhood of a_k, $k = 1,\ldots,N+1$.

For each point $s \in C_{N+1}$ we note that there are unique points $s_N \in C_N$, $s_{N-1} \in C_{N-1},\ldots,s_1 \in C_1$ such that

$$s = r_N(s_N), s_N = r_{N-1}(s_{N-1}),\ldots,s_2 = r_1(s_1).$$

We shall observe that

$$(7) \qquad |f(r_j(s_j)) - f(s_j)| < \frac{B\epsilon}{2}|a_j| \text{ for each } j = 1,2,\ldots,N.$$

To see this, fix a $j \in \{1, 2, \ldots, N\}$. Either $s_j p_j > 0$ or $s_j p_j < 0$. Considering the former situation first, we have

$$\begin{aligned} |f(r_j(s_j)) - f(s_j)| &= |f(2p_j - s_j) - f(s_j)| \\ &< \frac{2|s_j - p_j|B\epsilon}{16} \\ &< \frac{2(|a_j| + \delta^*)B\epsilon}{16} \\ &\leq \frac{4|a_j|B\epsilon}{16} \\ &= \frac{B\epsilon}{4}|a_j|, \end{aligned} \tag{8}$$

where we have used the fact that $s_j \in T(p_j, \epsilon, B)$. Next, if $s_j p_j < 0$, then

$$\begin{aligned} |f(r_j(s_j)) - f(s_j)| &= |f(2p_j + s_j) - f(s_j)| \\ &\leq |f(2p_j + s_j) - f(-s_j)| + |f(-s_j) - f(s_j)| \\ &< \frac{2|-s_j - p_j|B\epsilon}{16} + \frac{2|s_j|B\epsilon}{16} \\ &< \frac{4(|a_j| + \delta^*)B\epsilon}{16} \\ &\leq \frac{8|a_j|B\epsilon}{16} \\ &= \frac{B\epsilon}{2}|a_j|. \end{aligned} \tag{9}$$

This time we used that $s_j \in T(0, \epsilon, B)$ and $-s_j \in T(p_j, \epsilon, B)$. ¿From (8) and (9) we obtain the claim (7).

Hence, for each $s \in C_{N+1}$ we have

$$\begin{aligned} |f(s) - f(s_1)| &= |f(r_N(s_N) - f(s_1)| \\ &\leq \sum_{j=1}^{N} |f(r_j(s_j)) - f(s_j)| \\ &< \sum_{j=1}^{N} \frac{B\epsilon}{2}|a_j| \qquad \text{(from (7))} \\ &< \frac{B\epsilon}{2}\sum_{j=1}^{N}(1-\epsilon)^{j-1}a_1 \qquad \text{(from (3))} \\ &< \frac{B\epsilon a_1}{2}\sum_{j=1}^{\infty}(1-\epsilon)^{j-1} \end{aligned}$$

$$= \frac{Ba_1}{2}. \tag{10}$$

Consequently, for each $s \in C_{N+1}$ we have

$$\begin{aligned} f(s) &> f(s_1) - \frac{Ba_1}{2} \\ &> Ba_1 - \frac{Ba_1}{2} \\ &= \frac{Ba_1}{2}. \end{aligned}$$

This is an impossible situation, however, since $0 \in I_{N+1}$ and hence C_{N+1} is second category in $(-\delta^*, \delta^*)$, contradicting (2). This completes the proof of the lemma.

Theorem 1 *Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. The set of points of qualitative continuity of f where $f'_{qs}(x)$ exists and is finite, but $f'_q(x)$ does not exist is σ-$(1-\epsilon)$-symmetrically porous for any choice of $\epsilon \in (0,1)$.*

PROOF. The set we are interested in can be written as

$$\{x \in C_q(f) : -\infty < f'_{qs}(x) < Q^+f(x)\} \cup \{x \in C_q(f) : Q_+f(x) < f'_{qs}(x) < \infty\}$$

$$\cup \{x \in C_q(f) : -\infty < f'_{qs}(x) < Q^-f(x)\} \cup \{x \in C_q(f) : Q_-f(x) < f'_{qs}(x) < \infty\}$$

Let $0 < \epsilon < 1$. By taking into account $f(-x)$, $-f(-x)$ and $-f(x)$, we see it is enough to show that

$$S = \{x : -\infty < f'_{qs}(x) < Q^+f(x)\}$$

is σ-$(1-\epsilon)$-symmetrically porous.

Pick $z \in S$. For this z, let $r = r_z$ be a rational number such that

$$f'_{qs}(z) - \frac{\epsilon}{16}(Q^+f(z) - f'_{qs}(z)) < r < f'_{qs}(z).$$

This leads to

$$0 < f'_{qs}(z) - r < \frac{\epsilon}{16}(Q^+f(z) - f'_{qs}(z)) \le \frac{\epsilon}{16}(Q^+f(z) - r)$$

which gives us

$$\left|f'_{qs}(z) - r\right| < \frac{\epsilon}{16}(Q^+f(z) - r) \tag{11}$$

Now define the function $g_r(x) = f(x) - rx$. Then by using (11) we have

$$\frac{\epsilon}{16}Q^+(g_r(z)) > \left|(g_r)'_{qs}(z)\right|.$$

Thus for our set S we have $S \subseteq \cup\{S_\epsilon(g_r) : r \text{ is rational}\}$ where $S_\epsilon(g_r)$ is defined by Lemma 1. Applying Lemma 1 completes this proof.

We note here that this result cannot be viewed as a corollary of a strengthening of Theorem 4.1 in [4], because in Proposition 1 of [2], Evans constructs a monotone Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ such that the exceptional set for Theorem 4.1 of [4] is not σ-symmetrically porous.

In [1] Evans improved Zajíček's result in [9] to say the following: for an arbitrary $f : \mathbb{R} \to \mathbb{R}$ if the points of finite symmetric differentiability are contained in the closure of the points of continuity then the set of points of finite symmetric differentiability where f is not differentiable is σ- $(1-\epsilon)$-symmetrically porous. The following example will show that a qualitative version is not true.

Example 1 *There exists an* $f : \mathbb{R} \to \mathbb{R}$ *which is of Baire class one for which the set of points where* $|f'_{qs}(x)| < \infty$ *but* $f'_q(x)$ *does not exist has positive measure.*

PROOF. Let $C \subset [0,1]$ be a Cantor set of positive measure. Define f to be the characteristic function of C. Note that for any real number x, $f'_{qs}(x) = 0$. However, for any $x \in C$, $f'_q(x)$ does not exist.

4. A Measure Theoretic Analogue

In this section we will show that the previous results will hold if instead of using category as an indicator of size, we use the notion of measure. Let $f : \mathbb{R} \to \mathbb{R}$, let $x_0 \in \mathbb{R}$ and let μ and μ^* denote Lebesgue measure and outer measure respectively. The *measure limit superior from the right of* f *at* x_0 is defined as

$$\mu\text{-}\limsup_{x \to x_0+} f(x) = \inf\{y : \mu\{x : f(x) > y\} \text{ is zero in a right neighborhood of } x_0\}.$$

Similary we define *measure limit inferior from the right of* f *at* x_0, *measure limit inferior and superior from the left*, and if the inferior and superior limits agree we call it the *measure limit of* f *at* x_0. If $f(x_0) = \mu\text{-}\lim_{x \to x_0} f(x)$ then f is *measure continuous at* x_0, and we define

$$C_\mu(f) = \{x : f \text{ is measure continuous at } x\}.$$

The upper right measure derivate of f at x_0 is defined by

$$M^+ f(x_0) = \mu\text{-}\limsup_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h},$$

with $M^{-}f(x_0)$, $M_{+}f(x_0)$, and $M_{-}f(x_0)$ all defined in the obvious manner. If all four of the derivates are the same then we call this common value the *measure derivative of f at x_0* and denote it by $f'_{\mu}(x_0)$.

Finally, looking at symmetric measure derivatives we say the *upper measure symmetric derivate of f at x_0* is

$$\overline{M}^{s} f(x_0) = \mu\text{-}\limsup_{h\to 0+} \frac{f(x_0+h)-f(x_0-h)}{2h},$$

while the *lower measure symmetric derivate of f at x_0* is

$$\underline{M}^{s} f(x) = \mu\text{-}\liminf_{h\to 0+} \frac{f(x_0+h)-f(x_0-h)}{2h}.$$

If $\overline{M}^{s} f(x_0) = \underline{M}^{s} f(x_0)$, then the common value is called the *measure symmetric derivative of f at x_0* and is denoted $f'_{\mu s}(x_0)$.

Using these measure related ideas we have the following analogue of Lemma 1.

Lemma 1* *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function and let $0 < \epsilon < 1$. The set*

$$S_{\epsilon}(f) \equiv \left\{ x \in C_{\mu}(f) : |f'_{\mu s}(x)| < \frac{\epsilon M^{+} f(x)}{16} \right\}$$

is σ-$(1-\epsilon)$-symmetrically porous.

Only the obvious changes need be made in the proof given for Lemma 1. For example, we set

$$S_{\epsilon,B}(f) \equiv \left\{ x \in C_{\mu}(f) : |f'_{\mu s}(x)| < \frac{\epsilon B}{16} < M^{+} f(x) \frac{\epsilon}{16} \right\}$$

and redefine $S_{\epsilon,B,m}(f)$ to be

$$\left\{ x \in S_{\epsilon,B}(f) : \mu\left(T(x,\epsilon,B) \cap (x - \frac{1}{m}, x + \frac{1}{m}) \right) = \frac{2}{m} \right\}.$$

We pick an a_1 such that $\{x : f(x) > Ba_1\}$ intersects every open neighborhood of a_1 in a set of positive outer measure. Also, we choose a $\delta^{*} > 0$ such that

$$\{x \in (-\delta^{*}, \delta^{*}) : |f(x)| < Ba_1/2\} \text{ is of full measure in } (-\delta^{*}, \delta^{*}).$$

Finally, we note that for the sets C_k we know $\mu^{*}(C_{k+1}) = \mu^{*}(C_k)$.

As before, the following theorem readily follows from the lemma.

Theorem 1* *Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Consider the set of points of measure continuity of f where $f'_{\mu s}(x)$ exists and is finite, but $f'_{\mu}(x)$ does not exist. This set is σ-$(1-\epsilon)$-symmetrically porous for any choice of $\epsilon \in (0,1)$.*

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