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SOME RESULTS CONCERNING HAMEL BASES

Abstract

In this paper we show that under Martin's Axiom there is a Hamel basis H for which the set $E^+(H)$ of all finite linear combinations of elements from H with nonnegative rational coefficients is both of Lebesgue measure zero and of the first Baire category. We also obtain that the additive group of the reals $\mathbb{R}$is a direct sum of two subgroups one of which is uncountable, of measure zero and of the first Baire category.

1. Introduction

A subset H of the reals $\mathbb{R}$is called a Hamel basis, if H spans $\mathbb{R}$and H is linearly independent over the set of rationals. $E^+(H)$ is the set of all finite linear combinations of elements from H with nonnegative rational coefficients. Under the assumption of the continuum hypothesis, it was shown in [2, Th. 2] that there exists a Hamel basis H for $\mathbb{R}$such that $E^+(H)$ is a Lusin set (an uncountable set with the property that every uncountable subset of it is of second Baire category is called a Lusin set). H. Miller proved in [7, Th. 1](see also MR: 90 e 04002) that under the assumption of Martin's Axiom, there exists a Hamel basis H for $\mathbb{R}$such that $E^+(H)$ is a set of the first Baire category. He asks in [7] whether there exists a Hamel basis H for $\mathbb{R}$such that $E^+(H)$ is both of Lebesgue measure zero and of the first Baire category. The purpose of this paper is to show that under Martin's Axiom such a basis exists. We also obtain that the additive group of the reals $\mathbb{R}$is a direct sum of two subgroups G and H such that one of them is uncountable, of measure zero and of the first Baire category.

Key Words: **Martin's Axiom, Hamel Basis, Lebesgue measure zero, first Baire category, additive groups of real numbers**

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Notation: Throughout this paper, the set of all real numbers, the set of all rational numbers, the set of all integers, and the set of all positive integers are denoted by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. The initial ordinal number of the cardinal number of $\mathbb{R}$is denoted by Ω. For a subset A of $\mathbb{R}$, span A is the set of all finite linear combinations of elements from A with rational coefficients, $E^+(A)$ is the set of all finite linear combinations of elements from A with nonnegative rational coefficients, and rA is the set of all real numbers of the form ra, where $a \in A$. $A + B$ is the algebraic sum of subsets A and B of $\mathbb{R}$.

2. Results

Theorem 1 *Under the assumption of Martin's Axiom, there exists a Hamel basis H for $\mathbb{R}$such that $E^+(H)$ is both of (Lebesgue) measure zero, and of the first (Baire) category.*

PROOF. Assuming Martin's Axiom, there exists a measure zero, first category set A containing a translate of every set of cardinality less than the cardinality of the continuum [5]. Let $B = \cup_{q \in \mathbb{Q}} qA$. Then B is a measure zero, first category set satisfying the above mentioned translation property, and $qB = B \ \forall q \in \mathbb{Q}\backslash\{0\}$ (*). We are done if we show that there exists a Hamel basis H for $\mathbb{R}$such that $E^+(H) \subseteq B$. To do this, well-order $B = (b_\xi)_{\xi<\Omega}$, and construct two sequences $(c_\xi)_{\xi<\Omega}$ and $(d_\xi)_{\xi<\Omega}$ with the property "b_ξ is a linear combination of c_ξ and d_ξ with rational coefficients, and all finite linear combinations of elements from the two sequences with nonnegative rational coefficients are in B" as follows. Let $c_1 = b_1$ and $d_1 = 0$. Then $b_1 \in \text{span}\{c_1, d_1\}$, and $E^+\{c_1, d_1\} \subseteq B$. Suppose that $1 < \mu < \Omega$, and for each $\xi < \mu, c_\xi$ and d_ξ have been defined such that $b_\xi \in \text{span}\{c_\xi, d_\xi\}$, and $E^+(\{c_\xi : \xi < \mu\} \cup \{d_\xi : \xi < \mu\}) \subseteq B$. For simplicity, denote $\{c_\xi : \xi < \mu\} \cup \{d_\xi : \xi < \mu\}$ by H_μ. By (*), there exists $h \in R$ such that $E^+(H_\mu) + b_\mu\mathbb{Q} + h \subseteq B$. If we let $c_\mu = b_\mu + h$ and $d_\mu = h$, then it is not hard to see that $b_\mu \in \text{span}\{c_\mu, d_\mu\}$, and $E^+(H_\mu \cup \{c_\mu, d_\mu\}) \subseteq B$. Thus, by transfinite induction, we have constructed the sequences $(c_\xi)_{\xi<\Omega}$ and $(d_\xi)_{\xi<\Omega}$ with the desired properties. Let $K = \{c_\xi : \xi < \Omega\} \cup \{d_\xi : \xi < \Omega\}$, and let $\mathcal{F}$ be the family of all rationally independent subsets of K. Partially order $\mathcal{F}$ by set inclusion $\subseteq$. Then, by Zorn's lemma, $\mathcal{F}$ has a maximal element, say H. Clearly $E^+(H)$ is both of measure zero and of the first category. To show that H is a Hamel basis for $\mathbb{R}$, let $k \in K$ and $k \notin H$. Then, since $H \cup \{k\}$ is not rationally independent and H is rationally independent, k can be expressed as a finite linear combination of elements from H with rational coefficients. Hence span $H \supseteq K$. By our construction, span $K \supseteq B$, and consequently span $H \supseteq B$. Again by (*), for every $r \in R$, there exists $t \in R$ such that

$\{0, r\} + t \subseteq B$, and consequently $r = (r+t) - t \in B - B$. Hence span $H = R$, and since H is rationally independent, H is a basis for $\mathbb{R}$. □

P. Erdös proved in [1, Th. 1] that under the assumption of the continuum hypothesis, there are additive subgroups of the reals, which are of measure zero and of the second category; of the first category and not of measure zero. It was shown in [10] (see also MR: 91 j 26001) that under the assumption of the continuum hypothesis, there are additive subgroups of the reals, which are of the second category and nonmeasurable; of the second category without the Baire property. Without the assumption of the continuum hypothesis, the following theorem shows that the subgroups mentioned in [10] exist. It is interesting to compare part(3) of the following theorem with [3, Th. 5] and [4, Th. 1].

Remark 1 *It is well known that if A is a second category subset of $\mathbb{R}$with the Baire property, then $A - A$ contains a nonempty open interval (see [8] or [9]). Hence if G is a second category subgroup of $\mathbb{R}$with the Baire property, then $G = G - G$ contains a nonempty open interval, and consequently $G = R$. In other words, no second category proper subgroup of $\mathbb{R}$satisfies the Baire property.*

Theorem 2 *If H is a Hamel basis for $\mathbb{R}$, then*

1. *the Erdös set, $Z(H) = \{\sum_{i=1}^{n} m_i h_i : m_i \in \mathbb{Z}, h_i \in H, n \in \mathbb{N}\}$, is an additive subgroup of $\mathbb{R}$, which is everywhere of second category and non-measurable;*

2. *$Z(H)$ contains a measure zero, first category uncountable additive subgroup;*

3. *there are additive subgroups G and K of $\mathbb{R}$(indeed both are subspaces of R over the field of rationals) such that $R = G + K$, $G \cap K = \{0\}$, and G is both of measure zero and of the first category.*

PROOF.

1. The proof of this follows from the proof of Theorem 1 in [2] with appropriate minor modification.

2. Let A be a measure zero, first category set containing a translate of every countable set [6]. Let $B = \cup_{q \in \mathbb{Q}} qA$, and let $\mathcal{B}$ be the set of all additive subgroups, G, contained in B such that $\cup_{q \in \mathbb{Q}} qG = G$. Then $\mathcal{B}$ is preordered by set inclusion $\subseteq$, and by Zorn's lemma, $\mathcal{B}$ has a maximal element, say G. Clearly G is both of measure zero and of the first category. G is uncountable ; otherwise $G + r \subseteq B$ for some $r \in R$, and

since $qB = B$ and $qG = G$ $\forall q \in \mathbb{Q}\backslash\{0\}, B \supseteq G + \{qr : q \in \mathbb{Q}\}$ is an additive subgroup properly containing G, which contradicts that G is a maximal element of $\mathcal{B}$. Now let $G_n = \{g \in G : ng \in Z(H)\}$ for every $n \in \mathbb{N}$. To show that $G = \cup_{n\in\mathbb{N}} G_n$, let $g \in G$. Then there exist $a_i \in \mathbb{Z}$ and $b_i \in \mathbb{N}$ such that $g = \sum_{i=1}^{n} \frac{a_i}{b_i} h_i$ for some $n \in \mathbb{N}$. Let m be the least common multiple of $b_1, b_2, \ldots, b_n$. Then $mg \in Z(H)$, and consequently $g \in G_m$ and $G \subseteq \cup_{n\in\mathbb{N}} G_n$. Clearly $\cup_{n\in\mathbb{N}} G_n \subseteq G$. Hence for some n, nG_n is a measure zero, first category uncountable subgroup of $Z(H)$.

3. If follows from the proof of (2) that there is a measure zero, first category uncountable subgroup G of $\mathbb{R}$such that $\cup_{q\in\mathbb{Q}} qG = G$. Let $\mathcal{G}$ be the set of all additive subgroups, K, contained in $(R\backslash G) \cup \{0\}$such that $\cup_{q\in\mathbb{Q}} qK = K$. As in the proof of (2), $\mathcal{G}$ has a maximal element, say K. If $r \in R\backslash G$, then $qr+K$ is not contained in $(R\backslash G)\cup\{0\}$ for some $q \in \mathbb{Q}\backslash\{0\}$, and consequently $qr+k \in G$ for some $k \in K$; hence $r = \frac{qr+k}{q} - \frac{k}{q} \in G+K$.

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