Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 568–570

Bernd Kirchheim, Institute of Applied Mathematics, Comenius University, 842 15 Bratislava, Czechoslovakia &
Mathematical Institute, Kepler University, 4040 Linz-Auhof, Austria,
email :k312870@edvz.uni-linz.ac.at

THE SQUEEZING THEOREM IS INDEPENDENT

In [1] a system of six properties which completely characterizes the concept of convergence of real sequences was introduced. It consists of the following:

1.Definition *The collection of all real sequences will be denoted by S. The triple $(A, S, \equiv)$ is a convergence system on S provided that $A \subset S$, "$\equiv$" is a relation on A*[1)] *and:*

A1. *If $X = \{x_n\}_{n=1}^{\infty} \in A$ and $k \in \mathbb{R}$, then $kX = \{k x_n\}_{n=1}^{\infty} \in A$*
A2. *If $X \equiv Y, Z \equiv W$, and $X + Z, Y + W \in A$, then $Y + W \equiv X + Z$.*
A3. *If $X, Y \in A$, $Z \in S$, $x_n \leq z_n \leq y_n$ for all n and $X \equiv Y$, then $Z \equiv X$.*
A4. *If $X \in A$ and Y, Z are subsequences of X, then $Y \equiv Z$.*
A5. *$\{(-1)^n\}_{n=1}^{\infty} \notin A$.*
A6. *If $X \notin A$ is bounded, then X has subsequences $Y, Z \in A$ with $Y \not\equiv Z$.*

In [1] it was shown that the only convergence system on S is $(C, S, \equiv_0)$ where C is the family of all sequences converging in the classical sense (i.e. in the Euclidian topology) and $X \equiv_0 Y$ iff $\lim_{n\to\infty} x_n - y_n = 0$.

In the same paper it was asked whether Property A3 (the Squeezing Theorem) is independent of the other five properties. We will give the likely affirmative answer by providing an elementary construction of a system $(A, S, \equiv)$ fulfilling A1, A2, A4, A5, and A6 but such that $A \neq C$. According to the just mentioned result $(A, S, \equiv)$ can not satisfy A3.

2.Definition *Denote by A the system of all sequences $\{x_k\}_{k=1}^{\infty}$ of the*

Key Words: convergence systems, alternative definition of limits
Mathematical Reviews subject classification: Primary 26A03
Received by the editors September 22, 1992

1)In [1] "$\equiv$" is a relation on S but it is interesting only on A.

form

$$x_k = \alpha_1 2^{2^{n_k^1}} + \alpha_2 2^{2^{n_k^2}} + \cdots + \alpha_m 2^{2^{n_k^m}} + y_k \quad \text{for } k \geq 1, \tag{1}$$

where $\{y_k\}_{k=1}^{\infty}$ *is convergent in the Euclidean topology,* $\alpha_1, \ldots, \alpha_m \in I\!R$ *and each* $\{n_k^j\}_{k=1}^{\infty}$, $j \leq m$, *is a strictly increasing sequence of positive integers. We define for* $\{x_k\}_{k=1}^{\infty}$, $\{\tilde{x}_k\}_{k=1}^{\infty} \in A$ *that* $X \equiv \tilde{X}$ *iff* $\lim_{k\to\infty} y_k - \tilde{y}_k = 0$.

All we need in order to show that "$\equiv$" is correctly defined and that A1, A2, A4, A5, and A6 are satisfied is the next

3.Lemma *Let* $\{x_k\}_{k=1}^{\infty}$ *be as in* (1). *If* $\{x_k\}_{k=1}^{\infty}$ *is bounded, then* $x_k = y_k$ *except for finitely many* k.

Proof Otherwise, we can suppose (passing to a subsequence if necessary) that $y_k = 0 \neq x_k$ for all k. We consider the finite set

$$M = \{|\sum_{j\in J} \alpha_j|; J \subset \{1, \ldots, m\}\}$$

and denote $c = \min(M \setminus \{0\})$ and $C = m \cdot \max M$. Next, we take any $R > \sup_{k\geq 1} |x_k|$. Then there is a K such that

$$2^{2^{n_K^j - 1}} > 2\frac{C}{c}, 2\frac{R}{c} \text{ for all } \quad j \leq m. \tag{2}$$

We find

$$N = \max\{n; \sum_{j:\, n_K^j = n} \alpha_j \neq 0\}$$

since else

$$x_K = \sum_{n=1}^{\infty} \left(\sum_{j:\, n_K^j = n} \alpha_j \right) 2^{2^n} = 0.$$

Now, we estimate using (2) and $N = n_K^{j'}$ for some $j' \leq m$ that

$$\begin{aligned} |x_K| &\geq \ | \sum_{j:\, n_K^j = N} \alpha_j| \ 2^{2^N} - C\, 2^{2^{N-1}} \\ &\geq 2^{2^N} \cdot \frac{c}{2} + 2^{2^{N-1}} \left(\frac{c}{2} \cdot 2^{2^{N-1}} - C \right) \\ &\geq 2^{2^N} \cdot \frac{c}{2} > R, \end{aligned}$$

a contradiction finishing the proof.

Therefore, if there are two representations of a sequence in A

$$x_k = \alpha_1 2^{2^{n_k^1}} + \dots \alpha_m 2^{2^{n_k^m}} + y_k = \tilde{\alpha}_1 2^{2^{\tilde{n}_k^1}} + \dots \tilde{\alpha}_{\tilde{m}} 2^{2^{\tilde{n}_k^{\tilde{m}}}} + \tilde{y}_k$$

then the Lemma applied to the zero sequence

$$\{\alpha_1 2^{2^{n_k^1}} + \dots \alpha_m 2^{2^{n_k^m}} + (-\tilde{\alpha}_1) 2^{2^{\tilde{n}_k^1}} + \dots (-\tilde{\alpha}_{\tilde{m}}) 2^{2^{\tilde{n}_k^{\tilde{m}}}} + (y_k - \tilde{y}_k)\}_{k=1}^{\infty}$$

shows that $y_k = \tilde{y}_k$ except for finitely many k. Hence "$\equiv$" is well defined. Now A1, A2, and A4 are obvious. A5 follows since due to the Lemma any bounded sequence in A converges in the Euclidean topology. If X is bounded, then X contains subsequences X^+, X^- converging to $\limsup_{k\to\infty} x_k$ and $\liminf_{k\to\infty} x_k$, respectively. Hence, $X^+, X^- \in A$ and $X^+ \not\equiv X^-$ if $X \notin A$, i.e. not convergent. This shows A6. Finally, $\{2^{2^k}\}_{k=1}^{\infty} \in A \setminus C$ implies that our system does not fulfill the Squeezing theorem.

At the very end the author would like to express his opinion that despite the attractiveness of such alternative definitions for more experienced mathematicians the utility of this approach in first year calculus seems to be limited. Among others, due to the relatively complicated logical structure of the axiom system (from which the problem solved here arisen) and by the fact that it can lead the students to the impression that convergence is a very special topic basically related to linear or ordered spaces.

References

[1] **Peek, D.E.**; Limits without Epsilons; *Real Analysis Exchange, Vol. 17/2 (1991/92), 751—758.*