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A GENERALIZATION OF L'HÔPITAL'S RULE VIA ABSOLUTE CONTINUITY AND BANACH MODULES

Abstract

The main purpose of the present note is to make a first step in stating L'Hôpital's rule for functions of a real variable with higher-dimensional ranges. This is done by working with absolutely continuous functions in the framework of Banach modules.

1. Introduction.

Let us consider the version of L'Hôpital's rule which says that if f and g are two real-valued functions, everywhere differentiable on a real interval (a,b), if

$$\lim_{x\to x_0} g(x)=\infty, \quad \text{where } x_0=a \text{ or } x_0=b$$

and if $g'(x)\neq 0$ in a neighborhood of x_0, then

$$\lim_{x\to x_0} f'(x)/g'(x)=L$$

implies

$$\lim_{x\to x_0} f(x)/g(x)=L.$$

In the classical proof of this result, the following hypotheses seem to be crucial:

(i) the functions f,g are real-valued;

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(*ii*) f and g are everywhere differentiable in (a, b);

(*iii*) $g' \neq 0$ in a neighborhood of the limit point,

all, essentially, in applying Cauchy's theorem on finite increments (cf., e.g., [13, Th.5.13, p.94]).

During the last fifteen years, several authors have reconsidered or generalized in various ways the theorem above, see [2, 4, 5, 8, 9, 12, 20, 22]. For example, in [2] R.P.Boas gave a detailed discussion on a family of interesting counterexamples based on the relaxation of assumption (*iii*). On the other hand, A.M.Ostrowski [12] had previously made a first step in the weakening of assumptions (*ii*), (*iii*), stating a new version of L'Hôpital's theorem in the framework of ***absolutely continuous functions.*** Later Lee [8, 9], generalized the L'Hôpital-Ostrowski result via a strong weakening of (*ii*), (*iii*), based on the concepts of essential limit and approximate Peano derivative, and on various monotonicity theorems.

However, in spite of their deepness and completeness, all these results share the feature of considering ***only real-valued functions.*** Moreover, a strong use of the ordering of the range is made. Only recently [20], a new formulation of the rule has been proposed that does not resort to monotonicity assumptions.

The main purpose of the present note is just to generalize assumption (*i*), i.e. to make a first step in stating L'Hôpital's rule for functions of a real variable with ***higher-dimensional ranges.*** This is done by working with absolutely continuous functions, as in [12], so a weakening of (*ii*) and (*iii*) follows anyway. However, also in the case of real-valued functions, the present generalization has non-empty intersection and difference with those in [8, 9, 12, 20]. In particular, the function g' is allowed to be zero on ***positive*** measure subsets, and even to change sign, in each neighborhood of the limit point. This latter extension gives the tools for capturing some "cancellation phenomena", like those playing a central role in [2].

From the algebraic-topological viewpoint, we found it natural and convenient to state L'Hôpital's rule for functions f, g such that f is valued in a real (complex) Banach module X over a real (complex) Banach algebra A, and g is valued in the algebra A (about Banach modules see, e.g., [3]). The framework of absolutely continuous functions gives the following crucial possibility: we can work with ***the integral instead of the differential calculus.*** In infinite dimensional instances, as it is well-known, the connection of the integral to the differential calculus requires ***strong differentiability a.e.*** besides absolute continuity (cf. [7, Th.3.8.6, p.88]). In these cases, the ***Bochner integral*** [7, Ch.3] plays a key role in the generalization of L'Hôpital's rule. Abstract versions of the rule in the framework of ***vector-valued holomorphic functions***, based again on the possibility of integral representations, are given in [15]. An extension of the rule to the abstract case of the type "0/0" will appear in [16]. Recently the discrete analogue of L'Hôpital's rule, i.e. the well-known Cesaro's rule for real sequences, has also been generalized in various directions, cf. [10, 17].

In the next Section the main theorems are stated and proved. In Section 3, we shall give some examples and a detailed discussion on the counterexamples quoted above.

2. L'Hôpital's rule in the framework of Banach modules.

We shall prove the following:

Theorem 2.1 *Let X be a real (complex) right Banach module over the real (complex) Banach algebra with unit A, and (a,b) a real interval (bounded or unbounded).*

Let $f:(a,b)\to X$, $g:(a,b)\to A$ be strongly locally absolutely continuous (i.e. strongly absolutely continuous in any compact subinterval), and strongly differentiable a.e. in (a,b), with $g(x)\in Inv(A)$ $\forall x\in(a,b)$ and $\|(g(x))^{-1}\|_A\to 0$ as $x\to x_0, x_0=a$ or $x_0=b$.

Moreover suppose that there exists a Lebesgue measurable subset of (a,b), say E, such that f', g' are defined on E, $g'(x)\in Inv(A)$ $\forall x\in E$, x_0 is a limit point for E and:

(*I*)

$$\lim_{x\overset{E}{\to}x_0} f'(x)(g'(x))^{-1} = L\in X\ ; \tag{1}$$

(*II*) *for any fixed $\xi\in(a,b)$*

$$\limsup_{x\to x_0}\int_{E(\xi,x)}\|g'(t)(g(x))^{-1}\|_A dt<\infty\ , \tag{2}$$

where $E(\xi,x)=(\xi\wedge x,\xi\vee x)\cap E$;

(*III*) *defining $E^\star(\xi,x)=(\xi\wedge x,\xi\vee x)\cap E^c$, where E^c denotes the complement of E,*

$$\|\int_{E^\star(\xi,x)}(f'(t)-Lg'(t))(g(x))^{-1}dt\|_X\to 0\ \text{ as } x\to x_0\ . \tag{3}$$

Then

$$\lim_{x\to x_0} f(x)(g(x))^{-1}=L\ . \tag{4}$$

Before proving this theorem, we make some comments. First of all, note that Theorem 2.1 indeed represents a generalization of the classical result in the case of real-valued, absolutely continuous and everywhere differentiable functions. In fact if $g'(x) \neq 0$ in a neighborhood of x_0, say $V_{x_0} = (x_0 - \delta, x_0 + \delta) \cap (a, b)$, then by *Darboux' theorem* g' has constant sign in V_{x_0} and it is easily verified that hypotheses (II) and (III) hold, simply by choosing $E = V_{x_0}$. Observe that the left-hand side of (2) takes on the value 1, in this case.

As for (III), it is worth noting that it is automatically satisfied when *meas* $E^\star(\xi, x) = 0$ for x sufficiently close to x_0, or whenever the *cancellation*, $f'(t) = Lg'(t)$, occurs for all $t \in E^\star(\xi, x)$ (in particular when $f'(t)$ and $g'(t)$ both vanish in $E^\star(\xi, x)$).

Moreover, it is clear that a "left" version of Theorem 2.1 can be stated, and proved, in a similar way. Note that no requirement is made on the commutativity of A, or on the dimension of the spaces X and A. Obviously the strong differentiability a.e. hypothesis is essential only in infinite dimensional cases: then the integrals appearing in (III) and in the proof below are *Bochner integrals* (cf. [7, Ch.3]). We stress that the integral in (2) is finite for each x because the local Bochner integrability of g' is equivalent to the local integrability of $||g'||$ (cf., e.g., [7, Ch.3]).

Finally note that the hypothesis $||(g(x))^{-1}|| \to 0$ is equivalent to the classical one $||g(x)|| \to \infty$ only in the one-dimensional case, or in very special higher dimensional instances.

Proof. For convenience, we prove Theorem 2.1 in the case $x_0 = b$, the proof being similar for the limiting process $x \to a^+$. By the *absolute continuity* and the *strong differentiability a.e.* of f and g, for fixed $\xi \in (a, b)$, we can write:

$$f(x) = f(\xi) + \int_\xi^x f'(t)dt, \quad g(x) = g(\xi) + \int_\xi^x g'(t)dt \tag{5}$$

where the integrals on the r.h.s. are *Bochner integrals* (cf.[7, Ch.3]), and the obvious convention $\int_\xi^x = -\int_x^\xi$ if $x < \xi$ holds. We have

(6)

$$f(x)(g(x))^{-1} - L = \int_\xi^x [f'(t) - Lg'(t)]dt(g(x))^{-1} + [f(\xi) - Lg(\xi)](g(x))^{-1} ,$$

and the second additive term is infinitesimal as $x \to b^-$. Moreover

$$|| \int_\xi^x [f'(t) - Lg'(t)]dt(g(x))^{-1}||_X \leq || \int_{E(\xi,x)} (f'(t) - Lg'(t))(g(x))^{-1} dt||_X$$

$$+\left\| \int_{E^*(\xi,x)} (f'(t) - Lg'(t))(g(x))^{-1} dt \right\|_X \tag{7}$$

cf. (II) and (III), and the second additive term above is infinitesimal as $x \to b^-$, in view of (III).

Now, by (I) and (II), $\forall \varepsilon > 0 \ \exists x_1(\varepsilon) > \xi$ such that

$$\|f'(x)(g'(x))^{-1} - L\|_X < \varepsilon \text{ and}$$

$$\int_{E(\xi,x)} \|g'(t)(g(x))^{-1}\|_A dt \le K \ , \ \forall x \in (x_1(\varepsilon), b) \cap E \ , \tag{8}$$

for some positive constant K. Hence, recalling that $\exists M > 0 \ : \ \|m\alpha\|_X \le M\|m\|_X\|\alpha\|_A \ \forall \ m \in X, \ \alpha \in A$ (cf. [2, Ch.1, §9, def.12]), we get

$$\begin{aligned} &\left\| \int_{E(\xi,x)} [f'(t) - Lg'(t)] dt (g(x))^{-1} \right\|_X \\ &\le M \left\| \int_{(\xi,x_1(\varepsilon))\cap E} [f'(t) - Lg'(t)] dt \right\|_X \|(g(x))^{-1}\|_A \\ &+ M \int_{(x_1(\varepsilon),x)\cap E} \|f'(t)(g'(t))^{-1} - L\|_X \|g'(t)(g(x))^{-1}\|_A dt \end{aligned}$$

$$\le M \left\{ \psi(\varepsilon)\|(g(x))^{-1}\|_A + K\varepsilon \right\} \ , \ \forall x \in (x_1(\varepsilon), b) \tag{9}$$

again in view of (II), where

$$\psi(\varepsilon) = \left\| \int_{(\xi,x_1(\varepsilon))\cap E} [f'(t) - Lg'(t)] dt \right\|_X \tag{10}$$

is a nonnegative real function of ε.

The proof is completed by observing that $\exists x_2(\varepsilon) < b$ such that $\psi(\varepsilon)\|(g(x))^{-1}\|_A < \varepsilon \ \forall x \in (x_2(\varepsilon), b)$. **Q.E.D.**

For completeness, we state also the following result, which deals with a case not covered by Theorem 2.1, but which provides another natural generalization of L'Hôpital's rule in the framework of Banach algebras. The proof is completely analogous to the previous one.

Theorem 2.2 *Theorem* 2.1 *holds if $f : (a,b) \to K$, K being the field underlying the Banach algebra A ($K = R$ or $K = C$), and the only formal modification of writing $f'(t)e$, e denoting the unit element of A, instead of $f'(t)$, is made in (III).*

3. Examples and counterexamples.

3.1 Some examples in higher dimension.

We begin this Section by specializing the previous abstract results in two simple applications.

First of all, notice that the following two typical instances are included: X real (complex) Banach space, $A = R$ ($A = C$); X real (complex) Banach algebra, $A = X$. Regarding the first instance, we can state, as a corollary, the following result, which generalizes a well-known Abelian-type theorem (cf. [7, Thm.18.2.1, p.505]):

Corollary 3.1 *Let X be a real (complex) Banach space and $f : (a, +\infty) \rightarrow X$, where $a \geq 0$, a locally Bochner integrable function. If*

$$\lim_{x \to +\infty} \frac{f(x)}{x^p} = \lambda \in X \ , \ \ p > 0 \ , \tag{11}$$

then, $\forall \alpha \in (a, +\infty)$

$$\lim_{x \to +\infty} \frac{1}{x^{p+1}} \int_{\alpha}^{x} f(t)dt = \frac{\lambda}{p+1} \ . \tag{12}$$

The proof is immediate, since all hypotheses of Theorem 2.1 are readily verified taking as E the subset of $(a, +\infty)$ where the integral function in (12) is differentiable.

As for the second instance, i.e. the Banach algebras framework, let us discuss an application to the asymptotic theory of linear abstract and matrix differential equations. Consider the abstract differential equation

$$Y'' + Q(x)Y = 0 \ , \tag{13}$$

with $Q \in C^0([c, +\infty); A)$, $c \in R$, A denoting a given commutative C*-algebra with unit I, and let β be a *positive* element of A. If there exists a solution $Y_1(x)$ to (13) such that $Y_1(x) \sim \exp\{-\sqrt{\beta}x\}$, $Y_1'(x) \sim -\sqrt{\beta}\exp\{-\sqrt{\beta}x\}$ as $x \rightarrow +\infty$, $\sqrt{\beta}$ being the positive square root of β, then there is a solution $Y_2(x)$ such that $Y_2(x) \sim \exp\{\sqrt{\beta}x\}$, $Y_2'(x) \sim \sqrt{\beta}\exp\{\sqrt{\beta}x\}$. Hereafter $\exp\{\cdot\}$ will denote the exponential function in both the real and the Banach algebras frameworks.

First, it is easily verified that

$$Y_2(x) := 2\sqrt{\beta}Y_1(x) \int_{\alpha}^{x} [Y_1(t)]^{-2}dt \ , \tag{14}$$

solves (13) whenever $Y_1(x)$ does, α belonging to a suitable neighborhood of $+\infty$ where $Y_1(x)$ is invertible for each x. Such a neighborhood certainly exists in view of the asymptotic behavior of $Y_1(x)$. Now

$$\exp\{-\sqrt{\beta}x\}Y_2(x) \sim 2\sqrt{\beta}\exp\{-2\sqrt{\beta}x\}\int_\alpha^x [Y_1(t)]^{-2}dt$$

$$\sim \exp\{-2\sqrt{\beta}x\}[Y_1(x)]^{-2} \to I\ ,\quad x \to +\infty\ , \tag{15}$$

where L'Hôpital's rule has been applied as in Theorem 2.1 with $f(x) = \int_\alpha^x [Y_1(t)]^{-2}dt$ and $g(x) = \exp\{2\sqrt{\beta}x\}$. In fact,

$$\begin{aligned}\|(g(x))^{-1}\| &= \|\exp\{-2\sqrt{\beta}x\}\| = \rho(\exp\{-2\sqrt{\beta}x\}) \\ &= \exp\{-2x\min\sigma(\sqrt{\beta})\} = o(1)\ \text{as} x \to +\infty\end{aligned}$$

, where $\rho(\cdot)$ denotes the spectral radius and $\sigma(\cdot)$ the spectrum (cf. [14, Ch.10-11]). Moreover $\|g'(t)(g(x))^{-1}\| \leq 2\|\sqrt{\beta}\|\exp\{-2(x-t)\min\sigma(\sqrt{\beta})\}$, so that (II) holds the limsup there being bounded by $\|\sqrt{\beta}\|/\min\sigma(\sqrt{\beta})$. On the other hand (III) is trivial as $E = (a,b) = (\alpha,+\infty)$. Finally, simple manipulations and a further application of L'Hôpital's rule as in Theorem 2.1 show that $Y_2'(x)$ exhibits the prescribed asymptotic behavior as $x \to +\infty$.

A similar result can be obtained in the finite-dimensional non-commutative case of matrix differential equations, with $Q(x)$ a symmetric $n \times n$ matrix and β a symmetric positive definite matrix, under the additional requirement that $Y_1(x)$ commutes with β in the neighborhood of $+\infty$. In this case it is easily proved in view of its asymptotics that $Y_1(x)$ is a "prepared" solution to (13), since the "Wronskian" $W(x) := (Y_1)^T Y_1' - (Y_1')^T Y_1 \to 0$ as $x \to +\infty$, which implies that the matrix $W(x)$, being constant, must be identically zero, and hence

$$Y_2(x) := 2\sqrt{\beta}Y_1(x)\int_\alpha^x [Y_1^T(t)Y_1(t)]^{-1}dt \tag{16}$$

is a solution to (13) (cf. [6]). All proceeds then similarly to the previous case, by applying the generalized version of L'Hôpital's rule in Theorem 2.1. We observe, finally, that a solution like $Y_1(x)$ above certainly exists, e.g., in the classical cases $A = R$ or $A = C$ (cf. the Liouville-Green (WKB) approximation in [11]), and also in the abstract case (cf. [21]), under the assumption that $\int_a^{+\infty}\|Q(t)+\beta\|dt < \infty$.

To conclude this "higher-dimensional" subsection, we comment on the simple case $X = A = C$, which is included in a natural way in the present formulations of L'Hôpital's theorem, i.e. in both Theorems 2.1 and 2.2. Extensions of L'Hôpital's rule to complex-valued functions of a real variable seem to be missing in the literature. To the author's knowledge the only treatment, regarding however the underminate form "0/0", appears in [1]. The theorem proved there shows that the complex-valuedness leads to a further restriction on g': besides the usual condition $g' \neq 0$, it is required that g' take its values in an open sector of the complex plane with angle not exceeding π (two counterexamples are shown). On the contrary, such a restriction does not arise in Theorem 2.1 and 2.2, since the hypotheses on g' are perfectly compatible with the fact that its range is not contained in a sector as just described. Consider, e.g., $g(x) = exp((i+1)x)$, $f \equiv g$, $E = R = (a,b)$, $x_0 = +\infty$. Observe, finally, that in the complex case assumption (II) can be given a simple geometrical interpretation, whenever E is a full neighborhood of x_0, i.e.: the ratio between the length of the arc $t \mapsto g(t)$, $t \in [\xi \wedge x, \xi \vee x]$, and the length of the segment with the same end-points, $||g(x) - g(\xi)||$, must remain bounded as $x \to x_0$.

3.2 Reconsidering real-valuedness.

As announced in the Introduction, the present generalization of L'Hôpital's rule provides some new applications also in the classical instance of real-valued functions. Following Boas [2], consider the case

$$f'(x) = s(x)\psi(x) \ , \quad g'(x) = s(x)\omega(x) \tag{17}$$

where s vanishes in each neighborhood of the limit point, but $\lim \psi(x)/\omega(x)$ exists and is finite. We can refer to this situation as to a "cancellation phenomenon". The question is: when are we entitled to conclude that $\lim f/g = \lim \psi/\omega$? Actually, Theorem 2.1 (or Theorem 2.2) provides an extremely simple answer, whenever f and g are *absolutely continuous*.

In fact, if we choose $E = \{x : s(x) \neq 0\}$, hypotheses (I) and (III) are satisfied because $f'/g' = \psi/\omega$ in E and $f' = g' = 0$ in E^c. So we have only to check that (II) holds, which in this case is equivalent to the boundedness of $\int_\xi^x |g'(t)|dt/|\int_\xi^x g'(t)dt|$. In particular, it is sufficient that $g' \geq 0$ *or* $g' \leq 0$ a.e. in a neighborhood of x_0. Note, moreover, that $s(x)$, and hence $g'(x)$, is allowed to vanish on *positive* measure subsets in each neighborhood of the limit point. An occurrence of this type cannot be faced with the tools given in previous generalizations of L'Hôpital's rule.

In order to give a more detailed discussion, let us reconsider the family of conterexamples constructed in [2], generalizing an old idea by Stolz [18, 19]. We have

$$f(x) = \int_0^x [\lambda'(t)]^2 dt \ , \quad g(x) = f(x)\phi(\lambda(x)) \ , \tag{18}$$

where λ is periodic (not constant) with bounded derivative, ϕ is such that $\phi(\lambda(x))$ is bounded and both $\phi(\lambda(x))$ and $\phi'(\lambda(x))$ are bounded away from 0. Going back to the framework of the present paper, if ϕ is absolutely continuous in $[\min_{x\in R}\lambda(x), \max_{x\in R}\lambda(x)]$, a sufficient condition being for example the boundedness of ϕ' on such an interval, then $\phi \circ \lambda$ (and hence g) is absolutely continuous on the real line. In fact, f is by definition an absolutely continuous function. Notice that the conditions imposed on ϕ and on λ in [2], imply that $\lambda'(x)$ changes sign in each neighborhood of $+\infty$ (otherwise $\phi(\lambda(x)) \to \infty$, $x \to +\infty$), and that assumption (II) in Theorem 2.1 is not satisfied, as it can be easily checked.

The counterexample rests in the fact that applying the "cancellation rule" we obtain $\lim f'/g' = 0$, but $f/g = 1/\phi(\lambda)$ is bounded away from 0. In [2], Boas observed that if $\lambda'(x) \geq 0$ we obtain a correct application of the rule, since in this case $\phi(\lambda(x)) \to \infty$ as $x \to +\infty$.

Indeed, we stress that if $\lambda'(x) \geq 0$ we have $g' = f\lambda'\phi'(\lambda) + (\lambda')^2\phi(\lambda) \geq 0$, and hence assumption (II) in Theorem 2.1 holds, so we are effectively entitled to apply the cancellation rule.

Assumption (II) of Theorem 2.1, however, does not require that $g' \geq 0$ or $g' \leq 0$ a.e. in a neighborhood of the limit point. Consider for example $g(x) = \int_1^x g'(t)dt$ where

$$g'(x) = \begin{cases} 1 & \text{if } n \leq x < n + \frac{1}{2} \\ 0 & \text{if } n + \frac{1}{2} \leq x < n + \frac{3}{4} \\ -\frac{1}{n^2} & \text{if } n + \frac{3}{4} \leq x < n + 1 \end{cases} \tag{19}$$

for $n = 1, 2, \ldots$. Then one can check that hypothesis (II) in Theorem 2.1 still holds, choosing $E = (1, +\infty) \setminus \bigcup_{n=1}^{\infty} [n + 1/2, n + 3/4)$, essentially because the negative part of g' is integrable in $(1, +\infty)$. Note that g' changes sign and vanishes on positive measure subsets, in each neighborhood of $+\infty$.

We conclude observing that "cancellation phenomena" and other similar pathological instances arise and can be faced also in the general case, via the results of Section 2.

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