Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 553–556

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ON BOREL MEASURES ON SEPARABLE METRIC SPACES

It is known (consult, for example, [N, Exercise 8, page 88] or [R, 2.8, page 59]) that there exists a Lebesgue measurable set $E \subset [0,1]$ such that for any open set $U \subset (0,1)$, $m(U \cap E) > 0$ and $m(U \setminus E) > 0$, where m is Lebesgue measure. In a sense, both E and its complement are "locally large everywhere." This is usually proved by making E the union of countably many judiciously selected nowhere dense perfect sets with positive measure.

In this note we provide a constructive proof of the generalization:

Theorem 1. *Let m be a positive Borel measure on a separable metric space X such that $m(X) = 1$, and m vanishes on every countable set. Then there is an F_σ-set E such that for any open set U with $m(U) > 0$, we have*

$$m(U \cap E) > 0 \text{ and } m(U \setminus E) > 0.$$

Observe that we must have $m(P) = 0$ for any singleton set P in the hypothesis of Theorem 1. For otherwise we could fix $p_0 \in X$ and let $m(G) = 1$ if $p_0 \in G$ and $m(G) = 0$ if $p_0 \notin G$. The desired set E would not exist.

Moreover, we must have $m(X) < \infty$ in the hypothesis of Theorem 1. For otherwise we could let $m(G) = 0$ if G is a first Baire category set and $m(G) = \infty$ if G is a second Baire category set. Then E would not contain an interval, and hence $m(E) = 0$ contrary to the conclusion.

Theorem 1 applies, for example, to Lebesgue measure on a bounded subset of a Euclidean space. It also applies to Haar measure on compact subsets of some metric groups. For a discussion of Haar measure and for more examples, consult [M, section 17].

Proof of Theorem 1. Let V be the union of all the open subsets of X with measure 0. Now X is a separable metric space and hence X has a

Key Words: separable metric space, Borel measure, F_σ-set
Mathematical Reviews subject classification: Primary 28A12 Secondary 54D05
Received by the editors July 30, 1992

countable base. It follows that V is the union of countably many open sets of measure 0, and indeed $m(V) = 0$. Without loss of generality, we can substitute $X \setminus V$ for X in the proof. In other words we assume that $m(U) > 0$ for any nonvoid open set U.

First we let $(B(i_1))_{i_1 \geq 1}$ be a maximal family of mutually disjoint open balls of radius < 1 and diameter less than the diameter of X. (Of course there are countably many $B(i_1)$ because X is separable.)

In the second stage, for each $B(i_1)$ let $(B(i_1, i_2))_{i_2 \geq 1}$ be a maximal family of mutually disjoint open balls $B(i_1, i_2)$ of radius $< 1/2$ such that $m(B(i_1, i_2)) < 3^{-2} m(B(i_1))$ and $B(i_1, i_2) \subset B(i_1)$. (We can find such $B(i_1, i_2)$ because any singleton set is the intersection of a contracting sequence of open balls.) In the third stage, for each $B(i_1, i_2)$, let $(B(i_1, i_2, i_3))_{i_3 \geq 1}$ be a maximal family of mutually disjoint open balls $B(i_1, i_2, i_3)$ of radius $< 1/3$ such that $m(B(i_1, i_2, i_3)) < 3^{-3} m(B(i_1, i_2))$ and $B(i_1, i_2, i_3) \subset B(i_1, i_2)$.

In general, suppose the mutually disjoint balls $B(i_1, i_2, \ldots, i_{j-1})$ have been selected in the $(j-1)$-th stage. In the j-th stage we let

$$(B(i_1, i_2, \ldots, i_{j-1}, i_j))_{i_j \geq 1}$$

be a maximal family of mutually disjoint open balls $B(i_1, i_2, \ldots, i_{j-1}, i_j)$ of radius $< 1/j$ such that $m(B(i_1, i_2, \ldots, i_{j-1}, i_j)) < 3^{-j} m(B(i_1, i_2, \ldots, i_{j-1}))$ and $B(i_1, i_2, \ldots, i_{j-1}, i_j) \subset B(i_i, i_2, \ldots, i_{j-1})$. By induction the balls $B(\ldots)$ are selected in all stages from the first on up.

Let $B(i_1, \ldots, i_j)$ be a ball selected in the j-th stage. Put

$$
\begin{aligned}
(1)\quad Y(i_1, \ldots, i_j) \; = \; & B(i_1, \ldots, i_j, 1) \cup \bigcup_{i_{j+1} \geq 1} B(i_1, \ldots, i_j, i_{j+1}, 1) \\
\cup & \bigcup_{i_{j+1} \geq 1, i_{j+2} \geq 1} B(i_1, \ldots, i_j, i_{j+1}, i_{j+2}, 1) \\
\cup & \bigcup_{i_{j+1} \geq 1, i_{j+2} \geq 1, i_{j+3} \geq 1} B(i_1, \ldots, i_j, i_{j+1}, i_{j+2}, i_{j+3}, 1) \\
\cup & \cdots .
\end{aligned}
$$

Now,

$$m(B(i_1, \ldots, i_j, 1)) < 3^{-j-1} m(B(i_1, \ldots, i_j)).$$

But

$$m(B(i_1, \ldots, i_j)) \geq \sum_{i_{j+1} \geq 1} m(B(i_1, \ldots, i_j, i_{j+1})), \text{ and}$$

$$m(B(i_1, \ldots, i_j, i_{j+1}, 1)) < 3^{-j-2} m(B(i_1, \ldots, i_j, i_{j+1})).$$

It follows that

$$m\Big(\bigcup_{i_{j+1}} B(i_1,\ldots,i_j,i_{j+1},1)\Big) < 3^{-j-2}m(B(i_1,\ldots,i_j)).$$

It likewise follows that

$$m(B(i_1,\ldots,i_j,i_{j+1},i_{j+2},1)) < 3^{-j-3}m(B(i_1,\ldots,i_{j+1},i_{j+2})), \text{ and}$$

$$m\Big(\bigcup_{i_{j+1},i_{j+2}} B(i_1,\ldots,i_j,i_{j+1},i_{j+2},1)\Big) < 3^{-j-3}m(B(i_1,\ldots,i_j)).$$

Similarly,

$$m\Big(\bigcup_{i_{j+1},i_{j+2},i_{j+3}} B(i_1,\ldots,i_j,i_{j+1},i_{j+2},i_{j+3},1)\Big) < 3^{-j-4}m(B(i_1,\ldots,i_j))$$

and so forth.

From (1) we deduce that

$$m(Y(i_1,\ldots,i_j)) < (3^{-j-1}+3^{-j-2}+3^{-j-3}+\cdots)m(B(i_1,\ldots,i_j))$$

and

$$m(Y(i_1,\ldots,i_j)) < (3^{-j}/2)m(B(i_1,\ldots,i_j)) < m(B(i_1,\ldots,i_j)). \tag{2}$$

Put

$$E_1 = \bigcup_{j=1}^{\infty}\bigcup_{i_1,\ldots,i_{2j-1}} [B(i_1,\ldots,i_{2j-1},1)\setminus Y(i_1,\ldots,i_{2j-1},1)],$$

$$E_2 = \bigcup_{j=1}^{\infty}\bigcup_{i_1,\ldots,i_{2j}} [B(i_1,\ldots,i_{2j},1)\setminus Y(i_1,\ldots,i_{2j},1)].$$

The difference of two open sets is an F_σ-set, so both E_1 and E_2 are F_σ-sets. Moreover $B(i_1,\ldots,i_{2j-1},1)\setminus Y(i_1,\ldots,i_{2j-1},1)$ and

$$B(i_1,\ldots,i_{2k},1)\setminus Y(i_1,\ldots,i_{2k},1)$$

are disjoint sets for any j and k, and hence E_1 and E_2 are disjoint.

Now let U be a nonvoid open set. Let S_1 be an open ball of radius r such that the ball T with the same center and radius $4r$ lies in U, and $r < 1$. By the maximality of $(B(i_1))_{i_1}$, there is an index p_1 such that $S_1 \cap B(p_1)$ is nonvoid. Let S_2 be an open ball of radius $< 1/2$ such that $S_2 \subset S_1 \cap B(p_1)$ and $m(S_2) < 3^{-2}m(B(p_1))$. Again by maximality there is an index p_2 such that $S_2 \cap B(p_1, p_2)$ is nonvoid. Let S_3 be an open ball of radius $< 1/3$ such that $S_3 \subset S_2 \cap B(p_1, p_2)$ and $m(S_3) < 3^{-3}m(B(p_1, p_2))$. Again there is an index p_3 such that $S_3 \cap B(p_1, p_2, p_3)$ is nonvoid. By induction we produce a contracting sequence of open balls $S_1 \supset S_2 \supset \cdots \supset S_j \supset \cdots$ and indices $p_1, p_2, \ldots, p_j, \ldots$ such that the radius of S_j is less than $1/j$ and $S_j \cap B(p_1, p_2, \ldots, p_j)$ is nonvoid.

Let k be an index such that $1/(2k) < r$. Then $S_{2k} \subset S_1$ and $S_1 \cap B(p_1, \ldots, p_{2k})$ is nonvoid. But the radius of $B(p_1, \ldots, p_{2k})$ is less than $1/(2k)$ and less than r. Now S_1 and T are balls with the same center, S_1 has radius r and T has radius $4r$. It follows that $B(p_1, \ldots, p_{2k}) \subset T \subset U$. So U contains each of the sets $B(p_1, \ldots, p_{2k})$, $B(p_1, \ldots, p_{2k}, 1)$ and $B(p_1, \ldots, p_{2k}, 1, 1)$. Then $U \cap E_1$ contains the set $B(p_1, \ldots, p_{2k}, 1, 1) \setminus Y(p_1, \ldots, p_{2k}, 1, 1)$ and $U \cap E_2$ contains the set $B(p_1, \ldots, p_{2k}, 1) \setminus Y(p_1, \ldots, p_{2k}, 1)$. We deduce from (2) that $m(U \cap E_1) > 0$ and $m(U \cap E_2) > 0$.

Because E_1 and E_2 are disjoint F_σ-sets, we can select either E_1 or E_2 for E in Theorem 1. □

It is not difficult to prove that there is an infinite sequence of mutually disjoint F_σ-sets, each of which suffices for E in Theorem 1.

References

[M] M.E. Munroe, *Introduction to measure and integration*, Addison-Wesley, Cambridge, 1953.

[N] I.P. Natanson, *Theory of functions of a real variable*, vol. 1, Ungar, New York, 1955.

[R] Walter Rudin, *Real and Complex Analysis*, second edition, McGraw-Hill, New York, 1974.