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Valeriu Popa, Department of Mathematics, Higher Education Institute, 5500 Bacău, Rumania

Takashi Noiri, Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866 Japan

ON β-CONTINUOUS FUNCTIONS

Abstract

In this paper, the authors obtain several characterizations of β-continuity due to Abd El-Monsef et al. [1] and show that almost quasi-continuity in the sense of Borsik and Doboš [3] is equivalent to β-continuity.

1. Introduction

As a generalization of semi-continuity [6] and precontinuity [8], Abd El-Monsef et al. [1] defined β-continuous functions. Quite recently, Borsik and Doboš [3] have introduced the notion of almost quasi-continuity which is weaker than that of quasi-continuity [7] and obtained a decomposition theorem of quasi-continuity. In this paper, we obtain several characterizations of β-continuity and show that almost quasi-continuity is equivalent to β-continuity.

2. Preliminaries

Throughout the present paper, X and Y always mean topological spaces. Let A be a subset of a topological space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be **semi-open** [6] (resp. **preopen** [8], **β-open** [1] or **semi-preopen** [2]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A)), A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$). The family of all semi-open (resp. preopen, β-open) sets of X is denoted by $SO(X)$ (resp. $PO(X), \beta(X)$). The complement of a semi-open (resp. preopen, β-open) set is said to be **semi-closed** [4] (resp. **preclosed** [5], **β-closed** [1]). The intersection of all semi-closed (resp. preclosed, β-closed) sets containing a set A is called the **semi-closure** (resp. **preclosure**, **β-closure**) of A and is denoted by

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$s\mathrm{Cl}(A)$ (resp. $p\mathrm{Cl}(A), \beta\mathrm{Cl}(A)$). The union of all semi-open (resp. preopen, β-open) sets contained in A is called the **semi-interior** (resp. **preinterior**, **β-interior**) of A and is denoted by $s\mathrm{Int}(A)$ (resp. $p\mathrm{Int}(A), \beta\mathrm{Int}(A)$).

In [2], among others, the following properties were established:

Lemma 2.1 *(Andrijević [2]). Let A be a subset of X. Then*

(a) $\beta\mathrm{Cl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(b) $\beta\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

(c) $s\mathrm{Cl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(A))$,

(d) $s\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(A))$,

(e) $s\mathrm{Int}(\mathrm{Cl}(A)) = \mathrm{Cl}(p\mathrm{Int}(A)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,

(f) $s\mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(p\mathrm{Cl}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

3. Almost quasi-continuity and β-continuity

Definition 3.1 *A function $f : X \to Y$ is said to be* **β-continuous** *[1] if $f^{-1}(V) \in \beta(X)$ for every open set V of Y.*

Definition 3.2 *A function $f : X \to Y$ is said to be* **almost quasi-continuous** *[3] at a point x of X if for each neighborhood V of $f(x)$ and each neighborhood U of x, the set $f^{-1}(V) \cap U$ is not nowhere dense. A function is said to be* **almost quasi-continuous** *[3] if it has the property at every point of the domain.*

First, we obtain characterizations of almost quasi-continuous functions.

Theorem 3.3 *The following are equivalent for a function $f : X \to Y$:*

(a) *f is almost quasi-continuous at a point x of X.*

(b) *For each neighborhood V of $f(x)$ and each neighborhood U of x, there exists a nonempty open set G such that $G \subset U$ and $G \subset \mathrm{Cl}(f^{-1}(V))$.*

(c) *For each neighborhood V of $f(x)$, there exists $U \in SO(X)$ containing x such that $U \subset \mathrm{Cl}(f^{-1}(V))$.*

(d) *for each neighborhood V of $f(x)$, $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))$.*

(e) *for each neighborhood V of $f(x)$, there exists $U \in \beta(X)$ containing x such that $f(U) \subset V$.*

PROOF. (a) $\Rightarrow$ (b): Let U be any neighborhood of x and V any neighborhood of $f(x)$. Then $f^{-1}(V) \cap \mathrm{Int}(U)$ is not nowhere dense and hence we have

$$\emptyset \neq \mathrm{Int}(\mathrm{Cl}(f^{-1}(V) \cap \mathrm{Int}(U))) \subset \mathrm{Cl}(\mathrm{Int}(U)).$$

Put $G = \mathrm{Int}(\mathrm{Cl}(f^{-1}(V) \cap \mathrm{Int}(U))) \cap \mathrm{Int}(U)$. Then G is a nonempty open set such that $G \subset U$ and $G \subset \mathrm{Cl}(f^{-1}(V))$.

(b) $\Rightarrow$ (d): This is obvious.

(d) $\Rightarrow$ (a): For any neighborhood U of x and any neighborhood V of $f(x)$, we have $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))))$ and hence

$$\begin{aligned} \emptyset &\neq \mathrm{Int}(U) \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}V))) \\ &= \mathrm{Int}[\mathrm{Int}(U) \cap \mathrm{Cl}(f^{-1}(V))] \subset \mathrm{Int}(\mathrm{Cl}(U \cap f^{-1}(V))), \end{aligned}$$

where we used the fact that if G is open, then $G \cap \mathrm{Cl}(A) \subset \mathrm{Cl}(G \cap A)$ for every subset A. Therefore, $f^{-1}(V) \cap U$ is not nowhere dense.

(e) $\Rightarrow$ (c): For each neighborhood V of $f(x)$, there exists $U_0 \in \beta(X)$ containing x such that $f(U_0) \subset V$. Put $U = \mathrm{Cl}(U_0)$. Then we have $U \in SO(X)$ and $x \in U \subset \mathrm{Cl}(f^{-1}(V))$.

(c) $\Rightarrow$ (d): Let V be any neighborhood of $f(x)$. There exists $U \in SO(X)$ containing x such that $U \subset \mathrm{Cl}(f^{-1}(V))$ and we obtain

$$x \in U \subset \mathrm{Cl}(\mathrm{Int}(U)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))).$$

(d) $\Rightarrow$ (e): Let V be any neighborhood of $f(x)$. Then $x \in f^{-1}(V)$. Now put $U = \beta\mathrm{Int}(f^{-1}(V))$. Then it follows from Lemma 2.1 (b) that $x \in U \in \beta(X)$ and $f(U) \subset V$.

Remark 1 *In [3, Remark 2], it is stated without the proof that (a) and (b) in Theorem 3.1 are equivalent for each other. Moreover, the referee pointed out that Theorem 3.3 (d) was used in the definition of almost quasi-continuity by Neubrunnová and Šalát [9].*

Theorem 3.4 *A function $f : X \to Y$ is almost quasi-continuous if and only if it is β-continuous.*

PROOF. The proof follows from Theorem 3.3 (d).

Theorem 3.5 *The following are equivalent for a function $f : X \to Y$:*

(a) *f is β-continuous;*

(b) *for each subset A of X, $f(\beta\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$;*

(c) *for each subset B of Y, $\beta\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$;*

(d) *for each subset B of Y,* $f^{-1}(\mathrm{Int}(B)) \subset \beta\mathrm{Int}(f^{-1}(B))$*;*

(e) *for each open set V of Y,* $f^{-1}(V) \subset s\mathrm{Int}(\mathrm{Cl}(f^{-1}(V)))$*;*

(f) *for each open set V of Y,* $f^{-1}(V) \subset \mathrm{Cl}(p\mathrm{Int}(f^{-1}(V)))$*;*

(g) *for each subset B of Y,* $\mathrm{Int}(p\mathrm{Cl}(f^{-1}(B))) \subset f^{-1}(\mathrm{Cl}(B))$*;*

(h) *for each subset A of X,* $f(\mathrm{Int}(p\mathrm{Cl}(A))) \subset \mathrm{Cl}(f(A))$*;*

(i) *for each subset A of X,* $f(s\mathrm{Cl}(\mathrm{Int}(A))) \subset \mathrm{Cl}(f(A))$*;*

(j) *for each subset B of Y,* $s\mathrm{Cl}(\mathrm{Int}(f^{-1}(B))) \subset f^{-1}(\mathrm{Cl}(B))$.

PROOF. The proof is obtained by using Lemma 2.1 and is left as an exercise.

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