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A NOTE ON ABSOLUTE SUMMABILITY METHODS

In this paper a generalization of a theorem of Bor [2] has been proved.

1. Introduction

Let Σa_n be a given infinite series with partial sums s_n, and $u_n = na_n$. By z_n^α and t_n^α we denote the nth Cesarò means of order α ($\alpha > -1$) of the sequences (s_n) and (u_n), respectively. The series Σa_n is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |z_n^\alpha - z_{n-1}^\alpha|^k < \infty. \tag{1.1}$$

But since $t_n^\alpha = n(z_n^\alpha - z_{n-1}^\alpha)$ (see [5]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.2}$$

Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

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defines the sequence (T_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |\Delta T_{n-1}|^k < \infty, \tag{1.5}$$

where

$$\Delta T_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad (n \geq 1). \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

2.

It is known that the summability $|\bar{N}, p_n|_k$ and summability $|C, \alpha|_k$ are, in general, independent of each other. For $\alpha = 1$, Bor [2] has established a relation between the $|\bar{N}, p_n|_k$ and $|C, 1|_k$ summability methods by proving the following theorem.

Theorem 2.1 *Let (p_n) be a sequence of positive real constants such that as $n \to \infty$*

$$np_n \asymp P_n \quad (\text{that is } np_n O(P_n) \text{ and } P_n = O(np_n)). \tag{2.1}$$

If Σa_n is summable $|\bar{N}, p_n|_k$, then it is also summable $|C, 1|_k$, $k \geq 1$.

Notice that, to see the hypothesis (2.1) in Theorem 2.1 is satisfied by at least one $p_n \neq 1$, it is sufficient to take $p_n = n$ for all values of n.

In the present paper we shall prove the following theorem, which is a generalization of Theorem 2.1.

Theorem 2.2 *Let (p_n) be a sequence of positive real constants such that condition (2.1) of Theorem 2.1 is satisfied and let (T_n) be the $(\bar{N}, p_n)$ mean of the series Σa_n. If*

$$\sum_{n=1}^{\infty} (P_n/p_n)^{(2-\alpha)k-1} |\Delta T_{n-1}|^k < \infty. \tag{2.2}$$

then the series Σa_n is summable $|C, \alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$.

It should be noted that if we take $\alpha = 1$ in this theorem, then we get Theorem 2.1.

3.

We need the following lemma for the proof of our theorem.

Lemma 3.1 . See ([3]). *If $\alpha > -1$ and $\alpha - \beta > 0$, then*

$$\sum_{n=v}^{\infty} \frac{A_{n-v}^{\beta}}{nA_n^{\alpha}} = \frac{1}{vA_v^{\alpha-\beta-1}}, \tag{3.1}$$

where

$$A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} \simeq \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \ A_0^{\alpha} = 1 \text{ and } A_{-n}^{\alpha} = 0 \text{ for } n > 0. \tag{3.2}$$

4. Proof of Theorem 2.2

Let t_n^{α} be the nth (C, α) mean of the sequence (na_n), where $0 < \alpha \leq 1$. Then we have

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{4.1}$$

where A_n^{α} is as in (3.2). By (1.6), we have

$$a_n = -\frac{P_n}{p_n}\Delta T_{n-1} + \frac{P_{n-2}}{p_{n-1}}\Delta T_{n-2}. \tag{4.2}$$

Hence

$$\begin{aligned} t_n^{\alpha} &= \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v \left(-\frac{P_v}{p_v}\Delta T_{v-1} + \frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2} \right) \\ &= \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} (-v) A_{n-v}^{\alpha-1} \frac{P_v}{p_v}\Delta T_{v-1} - \frac{nP_n}{p_n A_n^{\alpha}}\Delta T_{n-1} \\ &\quad + \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} v A_{n-v}^{\alpha-1} \frac{P_{v-2}}{p_{v-1}}\Delta T_{v-2} \\ &= -\frac{nP_n}{p_n A_n^{\alpha}}\Delta T_{n-1} + \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} (-v) A_{n-v}^{\alpha-1} \frac{P_v}{p_v}\Delta T_{v-1} \\ &\quad + \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} (v+1) A_{n-v-1}^{\alpha-1} \frac{P_{v-1}}{p_v}\Delta T_{v-1} \end{aligned}$$

$$= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{\Delta T_{v-1}}{p_v}\left(-vP_v A_{n-v}^{\alpha-1} + (v+1)A_{n-v-1}^{\alpha-1}P_{v-1}\right).$$

Since

$$-vP_v A_{n-v}^{\alpha-1} + (v+1)P_{v-1}A_{n-v-1}^{\alpha-1} = -vP_v \Delta A_{n-v}^{\alpha-1} - vp_v A_{n-v-1}^{\alpha-1} + P_{v-1}A_{n-v-1}^{\alpha-1},$$

we have

$$\begin{aligned} t_n^\alpha &= -\frac{nP_n}{p_n A_n^\alpha}\Delta T_{n-1} - \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} v\frac{P_v}{p_v}\Delta A_{n-v}^{\alpha-1}\Delta T_{v-1} \\ &- \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}\Delta T_{v-1} + \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\frac{P_{v-1}}{p_v}A_{n-v-1}^{\alpha-1}\Delta T_{v-1} \\ &= t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{1}{n}|t_{n,r}^\alpha|^k < \infty, \quad \text{for } r = 1,2,3,4, \quad \text{by (1.2).} \tag{4.3}$$

Firstly, we have

$$\begin{aligned} \sum_{n=1}^{m}\frac{1}{n}|t_{n,1}^\alpha|^k &= \sum_{n=1}^{m} n^{k-1}(P_n/p_n)^k (A_n^\alpha)^{-k}|\Delta T_{n-1}|^k \\ &= O(1)\sum_{n=1}^{m} n^{k-1}(P_n/p_n)^k\, n^{-\alpha k}|\Delta T_{n-1}|^k \\ &= O(1)\sum_{n=1}^{m}(P_n/p_n)^{(2-\alpha)k-1}|\Delta T_{n-1}|^k = O(1) \end{aligned}$$

as $m \to \infty$, by virtue of the hypotheses.

Now, when $k > 1$, applying Hölder's inequality, with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\sum_{n=1}^{m+1}\frac{1}{n}|t_{n,2}^\alpha|^k \le \sum_{n=2}^{m+1}\frac{1}{n(A_n^\alpha)^k}\left(\sum_{v=1}^{n-1} v\frac{P_v}{p_v}|\Delta A_{n-v}^{\alpha-1}|\,|\Delta T_{v-1}|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{\left(\sum_{v=1}^{n-1} v^k\left(\frac{P_v}{p_v}\right)^k|\Delta A_{n-v}^{\alpha-1}|\,|\Delta T_{v-1}|^k\right)\left(\sum_{v=1}^{n-1}|\Delta A_{n-v}^{\alpha-1}|\right)^{k-1}}{n^{1+\alpha k}}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1} \frac{\left(\sum_{v=1}^{n-1} v^k \left(\frac{P_v}{p_v}\right)^k (n-v)^{\alpha-2}|\Delta T_{v-1}|^k\right)\left(\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\right)^{k-1}}{n^{1+\alpha k}} \\
&= O(1)\sum_{v=1}^{m} v^k \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-2}}{n^{1+\alpha k}}, \text{ when } (0<\alpha<1) \\
&= O(1)\sum_{v=1}^{m} v^{k-\alpha k-1} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} (n-v)^{\alpha-2} \\
&= O(1)\sum_{v=1}^{m} v^{k-\alpha k-1} \left(\frac{P_v}{p_v}\right)^k |\Delta T_{v-1}|^k.
\end{aligned}
$$

Thus when $0 < \alpha < 1$, we have

$$
\sum_{n=2}^{m+1} \frac{1}{n}|t_{n,2}^{\alpha}|^k = O(1)\sum_{v=1}^{m}(P_v/p_v)^{(2-\alpha)k-1}|\Delta T_{v-1}|^k = O(1) \text{ as } m \to \infty,
$$

by virtue of the hypotheses.

Remark 1 *It should be noted that when $\alpha = 1$, the summation equals zero as $\Delta A_{n-v}^{\alpha-1} = 0$.*

Again using Lemma 3.1, we get

$$
\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n}|t_{n,3}^{\alpha}|^k &\le \sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k}\left\{\sum_{v=1}^{n-1} vA_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{nA_n^{\alpha}}\left\{\sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right\} \times \left\{\sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{nA_n^{\alpha}}\sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{A_{n-v}^{\alpha-1}}{nA_n^{\alpha}}
\end{aligned}
$$

$$= O(1)\sum_{v=1}^{m} v^k|\Delta T_{v-1}|^k \frac{1}{v}.$$

Since $1-\alpha > 0$ and $k \geq 0$, we have $a^{(1-\alpha)k} \leq v^{(1-\alpha)k}$. Hence as in $t_{n,1}^{\alpha}$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n}|t_{n,3}^{\alpha}|^k &= O(1)\sum_{v=1}^{m} v^{k-1}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m} v^{k-1}v^{(1-\alpha)k}|\Delta T_{v-1}|^k \\
&= O(1)\sum_{v=1}^{m}(P_v/p_v)^{(2-\alpha)k-1}|\Delta T_{v-1}|^k = O(1) \text{ as } m\to\infty,
\end{aligned}$$

by virtue of the hypotheses.

Finally, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{|t_{n,4}^{\alpha}|^k}{n} &\leq \sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k}\left(\sum_{v=1}^{n-1} \frac{P_{v-1}}{p_v} A_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right)^k \\
&= \sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k}\left(\sum_{v=1}^{n-1} \frac{P_{v-1}}{P_v}\cdot\frac{P_v}{p_v} A_{n-v-1}^{\alpha-1}|\Delta T_{v-1}|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k}\left(\sum_{v=1}^{n-1} \frac{P_v}{p_v} A_{n-v}^{\alpha-1}|\Delta T_{v-1}|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{\left(\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right)\left(\sum_{v=1}^{n-1}\frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}\right)^{k-1}}{nA_n^{\alpha}} \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{nA_n^{\alpha}}\left(\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k A_{n-v}^{\alpha-1}|\Delta T_{v-1}|^k\right) \\
&= O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{A_{n-v}^{\alpha-1}}{nA_n^{\alpha}} \\
&= O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k \frac{1}{v},
\end{aligned}$$

by the lemma. Hence, as in $t^{\alpha}_{n,3}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\frac{1}{n}|t^{\alpha}_{n,4}|^k &= O(1)\sum_{v=1}^{m}(P_v/p_v)^k|\Delta T_{v-1}|^k v^{(1-\alpha)k}v^{-1}\\ &= O(1)\sum_{v=1}^{m}(P_v/p_v)^{(2-\alpha)k-1}|\Delta T_{v-1}|^k\\ &= O(1) \text{ as } m\to\infty.\end{aligned}$$

by virtue of the hypotheses. Therefore, we get that

$$\sum_{n=1}^{m}\frac{1}{n}|t^{\alpha}_{n,r}|^k = O(1) \quad\text{as}\quad m\to\infty,\quad\text{for}\quad r=1,2,3,4.$$

This completes the proof of the theorem.

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