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TYPICAL PROPERTIES OF CONTINUOUS FUNCTIONS VIA THE VIEROTIS TOPOLOGY

If E is a residual subset of a complete metric space X, each x in E is called a *typical element* of X. When the set of all elements of X with a given property is residual, we say that this property is *typical*. We consider typical properties in $C(X)$ where X is compact metrizable and $C(X)$ denotes the Banach space of all continuous real-valued functions with the norm $||f|| = \sup\{|f(x)| : x \in X\}$. In particular, we take $X \subseteq I = [0, 1]$.

Several typical properties in $C(I)$ are known (see [Br]). Among them, those concerning level sets $[f = a] = \{x : f(x) = a\}$, $a \in \mathbb{R}$, and, more generally, sets of the form $[f = h] = \{x : f(x) = h(x)\}$, $h \in D$, where D is a fixed subset of $C(I)$, have been studied. It turns out that each level set of a typical $f \in C(I)$ is small in various senses. We employ the Vietoris topology on the space $K(X)$ of all nonempty, closed sets in X and describe a new technique of showing that, for some $D \subseteq C(X)$ and $E \subseteq K(X)$, the set

$$\{f \in C(X) : (\forall h \in D)\,[f = h] \in E\}$$

is residual. We give two applications.

Let X be a compact metrizable space. Let $K(X)$ denote the space of all nonvoid closed subsets of X, with the *Vietoris topology* generated by the subbase consisting of sets

$$U(G) = \{F \in K(X) : F \subseteq G\}, \qquad V(G) = \{F \in K(X) : F \cap G \neq \emptyset\}$$

where G is an arbitrary open set in X. Then $K(X)$ is compact and metrizable ([Kr], §42 I, II). If Y is a topological space, $\psi : Y \to K(X)$ is called *upper semicontinuous* (abbr. *usc*) when $\psi^{-1}[U(G)]$ is open for each open $G \subseteq X$ (cf. [Kr], §43 I).

We say that $E \subseteq K(X)$ is of *type* G_δ^* (or is a G_δ^* *set*) if $E = \bigcap_{n\in\omega} W_n$ and each W_n is the union of sets of the form $U(G)$ for an open $G \subseteq X$.

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Note that some open sets of the form $V(G)$ are not of type G_δ^*. Indeed, consider any open G such that $\emptyset \neq G \neq X$. Suppose that $V(G)$ is of type G_δ^*, i.e. $V(G) = \bigcap_{n\in\omega} \bigcup_{t\in T_n} U(G_t^n)$ where T_n's are arbitrary sets and G_t^n's are open in X. If $F = \{x_1, x_2\}$ where $x_1 \in G$ and $x_2 \notin G$, then, obviously $F \in V(G)$. Hence, for each $n \in \omega$, there is some t_n in T_n such that $F \in U(G_{t_n}^n)$. Consequently, $\{x_2\} \in \bigcap_{n\in\omega} U(G_{t_n}^n) \subseteq V(G)$, which contradicts $x_2 \notin G$.

Proposition 1 *If $E \subseteq K(X)$ is a G_δ^* set, then, for any $D \subseteq C(X)$, the set*

$$\Omega(D, E) = \{f \in C(X) : (\forall h \in D)[f = h] \in E\}$$

is of type G_δ.

PROOF. For any $h \in D$, let $\sigma_h : C(X) \to C(X)$ and $\tau : C(X) \to K(X)$ be given by $\sigma_h(f) = f - h$ and $\tau(f) = f^{-1}[\{0\}]$, respectively. Then τ is *usc* (see [Kr], §43 I, Th.1) and σ_h is obviously continuous. Thus $\tau\circ\sigma_h$ is *usc*. Hence the mapping $\psi : C(X) \to K(X)$ given by $\psi(f) = \bigcap_{h\in D}(\tau\circ\sigma_h)(f)$ is also *usc* ([Kr], §43 I, Th. 5). This, by the definition of a G_δ^* set, easily implies that $\psi^{-1}[E]$ is of type G_δ. Since $\psi^{-1}[E] = \Omega(D, E)$, we get the assertion. □

For $X \subseteq I$ and $D \subseteq C(I)$, we denote $\{f|X : f \in D\}$ by $D|X$.

Theorem 1 *Let X be a closed subset of I. Let A denote the family of all analytic functions on I. If $E \subseteq K(X)$ is a G_δ^* set containing all finite sets in X, then, for any $D \subseteq A$ such that $A\backslash D$ is dense in $C(I)$, the set $\Omega(D|X, E)$ is dense of type G_δ, hence residual in $C(X)$.*

PROOF. In the light of Proposition 1, it suffices to show the density of $\Omega(D|X, E)$. This will be done if we prove $(A\backslash D)|X \subseteq \Omega(D|X, E)$ since, clearly, $(A\backslash D)|X$ is dense in $C(X)$. So, take arbitrary $f \in (A\backslash D)|X$ and $h \in D|X$. Then $[f = h]$ is finite since, otherwise, it has a point of accumulation and from f, $h \in A|X$ we would get $f = h$, a contradiction. So, $[f = h] \in E$ and, consequently, $f \in \Omega(D, E)$. □

The sets of all standard polynomials and of all trigonometric polynomials serve as examples of D in Theorem 1, which follows from the fact that they are dense in $C(I)$ and their intersection (the set of constant functions) is nowhere dense.

Note that the family of all finite sets in X is dense in $K(X)$ ([Kr], §17 II, Th.4), so a G_δ^* set E containing all finite sets in residual in $K(X)$. By Theorem 1, for the respective D, the operation $\Omega(D|X, \cdot)$ transforms E onto another residual set (in $C(I)$).

We are interested in the case when E in Theorem 1 consists of small sets, for instance, when it forms an ideal of closed sets (thus the assumption that E contains all finite sets becomes natural). Note that σ-ideals of compact sets were extensively studied in [KLW]. Now, among G_δ σ-ideals in $K(X)$, one may separate a new class of G_δ^* σ-ideals. Let us have a look at two examples.

Recall that a function $\gamma : \mathcal{P}(X) \to [0, +\infty)$ is a *Choquet capacity* on X if

(i) $\gamma(\emptyset) = 0$, and $A \subseteq B$ implies $\gamma(A) \le \gamma(B)$,

(ii) $\gamma(\bigcup_n A_n) = \sup_n \gamma(A_n)$ if $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$,

(iii) $\gamma(\bigcap_n F_n) = \inf_n \gamma(F_n)$ if $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ are compact sets,

(cf. [D], [KLW]). From (i) and (iii) it follows that if $F \in K(X)$ and $\gamma(F) < t$, there is an open $U \supseteq F$ such that $\gamma(U) < t$. So, the family of all closed γ-null sets forms a G_δ^* set in $K(X)$ and, if γ is subadditive, we get a G_δ^* σ-ideal (see [KLW], p. 280).

Corollary 1 *Let γ be a Choquet capacity on a closed $X \subseteq I$, vanishing on all finite sets. For a typical $f \in C(X)$, if $h \in D$ where $D \subseteq A$ and $A \setminus D$ is dense in $C(X)$, then $\gamma[f = h] = 0$.*

In particular, Corollary 1 works for any finite continuous Borel measure μ on X (this result is probably known) since the outer measure μ^* forms a Choquet capacity.

Example 1 (cf. [My]). Let $2^{<\omega}$ and 2^ω denote the sets of all finite and infinite sequences with terms in $\{0,1\}$, respectively. Assume that 2^ω is endowed with the product topology. For $B \subseteq 2^\omega$ and $L \subseteq \omega$, consider the following game $\Gamma(B, L)$ between two players: I and II. They choose the consecutive terms of $x \in 2^\omega$: player I picks $x(i)$ if $i \in \omega \setminus L$ and player II — if $i \in L$. When each of them makes his choice, he knows all previous moves. Player I wins if $x \in B$ and player II — if $x \notin B$. Let $V_{II}(L)$ denote the family of all B's for which player II has a winning strategy in $\Gamma(B, L)$. Now, consider a family of sets $L_s \subseteq \omega$, for $s \in 2^{<\omega}$, such that $L_{s0} \cup L_{s1} \subseteq L_s$ and $L_{s0} \cap L_{s1} = \emptyset$ for $s \in 2^{<\omega}$. Then $M = \bigcap\{V_{II}(L_s) : s \in 2^{<\omega}\}$ forms a σ-ideal called a *Mycielski ideal*. It is curious that there exists a set in M whose complement is of the first category and of measure zero (we mean the standard product measure on 2^ω, isomorphic to the Lebesgue measure on I).

From Proposition 2.1 in [Ba R] it follows at once that $M \cap K(2^\omega)$ is of type G_δ^*. Since 2^ω is homeomorphic to the classical Cantor set in I, we may assume that $2^\omega \subseteq I$ and from Theorem 1 we derive

Corollary 2 *For a typical f in $C(2^\omega)$, if $h \in D$, where $D \subseteq A$ and $A \setminus D$ is dense in $C(I)$, then $[f = h] \in M$.*

Let us compare the above results with similar theorems in the literature. The assumptions on D in Theorem 1 seem rather strong. For instance, from the theorem of Goffman ([GP], p. 158) it follows that, for any σ-compact $D \subseteq C(I)$, the set $\{f \in C(I) : (\forall h \in D)(\lambda[f = h] = 0)\}$ is residual (λ stands for the Lebesgue measure). A similar result concerning Hausdorff measures, where D is relatively large ($D \supsetneq A$), is announced in [H]. On the other hand, our Corollary 1 deals with any capacity (but $D \subsetneq A$). The role of Theorem 1 consists in its universal nature: it includes typical properties connected with each $E \subseteq K(X)$ of type G_δ^* containing finite sets (however, to ensure the density of $\Omega(D, E)$, we had to restrict the class of sets D). Note that D cannot be too large. The result of [BrH] states that, for any σ-compact $D \subseteq C(I)$, if E consists of closed bilaterally strongly porous sets, then $\Omega(D, E)$ is residual. (By the way, we know from [L] that all closed bilaterally strongly porous sets form a G_δ set in $K(I)$, but we do not know whether it is of type G_δ^*, so, Theorem 1 is useless at this moment.) On the other hand, Buczolich proved in [Bu] that, for each $f \in C(I)$, there is an absolutely continuous h such that $[f = h]$ is not bilaterally strongly porous. Thus $D =$ (absolutely continuous functions) is too large to make the respective $\Omega(D, E)$ residual.

Finally, note that, having a residual set H in $C(X)$, one can create the corresponding residual sets in $C(X) \times C(X)$. For instance, the set

$$H^* = \{\langle f, g\rangle \in C(X) \times C(X) : f - g \in H\}$$

is good. Indeed, consider a G_δ residual set $B \subseteq H$. Then B^* is also of type G_δ, by the continuity of $\langle f, g\rangle \mapsto f - g$. Moreover, for any $g \in C(X)$, the set $(B^*)_g = \{f \in C(X) : \langle f, g\rangle \in B^*\}$ equals $B + g = \{h + g : h \in B\}$, thus it is residual. Hence, by the Kuratowski-Ulam theorem, B^* is residual and H^* is residual, too.

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