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ON NECESSARY CONDITIONS FOR HENSTOCK INTEGRABILITY

Lu [1] proved the following Lemma:

Lemma 1 *If f is Henstock integrable on $[a,b]$, then there is a sequence $\{X_n\}$ of closed subsets of $[a,b]$ such that $X_n \subset X_{n+1}$ for all n, $[a,b]\setminus\bigcup_{n=1}^{\infty} X_n$ is of measure zero, f is Lebesgue integrable on each X_n and*

$$\text{(1)} \qquad \lim_{n\to\infty}(L)\int_{X_n} f(t)\,dt = (H)\int_a^b f(t)\,dt.$$

In this paper, we shall improve Lu's result so that "$[a,b]\setminus\bigcup_{n=1}^{\infty} X_n$ is of measure zero" is replaced by "$\bigcup_{n=1}^{\infty} X_n = [a,b]$" and (1) by

$$\text{(2)} \qquad \lim_{n\to\infty}(L)\int_{X_n\cap[a,x]} f(t)\,dt = (H)\int_a^x f(t)\,dt$$

uniformly on $[a,b]$. Furthermore, we give an equivalent definition of the Henstock integral, and a Convergence theorem. We remark that Nakanishi [2; p.81] proved an intermediate result, namely, (2) holds pointwise instead of uniformly on $[a,b]$.

First, we give some preliminaries (see [3]).

A function f is said to be Henstock integrable to A on $[a,b]$ if for every $\varepsilon>0$ there is a function $\delta(\xi)>0$ such that for any δ-fine division $D=\{([u,v],\xi)\}$ of $[a,b]$ we have

$$|(D)\sum f(\xi)(v-u)-A|<\varepsilon.$$

The following Henstock lemma [3; p.12] will be used.

Lemma 2 *If f is Henstock integrable on $[a,b]$ with the primitive F, then for every $\varepsilon>0$ there is a $\delta(\xi)>0$ such that for any δ-fine partial division $D'=\{([u,v],\xi)\}$ of $[a,b]$ we have*

$$|(D')\sum\{f(\xi)(v-u)-(F(v)-F(u))\}|<\varepsilon.$$

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Let $X \subset [a,b]$. A function F defined on $[a,b]$ is said to be $AC^*(X)$ if for every $\varepsilon > 0$ there is an $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $\{[a_k, b_k]\}$ with the endpoints $a_k, b_k \in X$ for all k satisfying

$$\sum_k |b_k - a_k| < \eta \text{ we have } \sum_k \omega(F; [a_k, b_k]) < \varepsilon$$

where ω denotes the oscillation of F over $[a_k, b_k]$. A function F is said to be ACG^* on $[a,b]$ if $[a,b]$ is the union of a sequence of subsets $\{X_i\}$ such that the function F is $AC^*(X_i)$ for each i.

It is well-known [3; p.34 and p.21] that if f is Henstock integrable on $[a,b]$ with the primitive F, then F is ACG^* on $[a,b]$ and $F'(x) = f(x)$ almost everywhere in $[a,b]$. Furthermore since F is continuous on $[a,b]$, [3; p.12] we may choose X_i, $i = 1, 2, \ldots$, in the definition of ACG^* to be closed sets. This fact will be used in the proof of Theorem 3 below.

Now we give the following definition.

Definition 1 *Let $\{X_n\}$ be a sequence of closed subsets of $[a,b]$ with $X_n \subset X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = [a,b]$. A function f defined on $[a,b]$ is said to fulfill the condition (L) on $\{X_n\}$ if f is Lebesgue integrable on each X_n and $(L)\int_{X_n \cap [a,x]} f(t)\,dt$ converge uniformly on $[a,b]$. Also, f is said to fulfill the condition (H) on $\{X_n\}$ if for each n there exists $\delta_n(\xi) > 0$ satisfying $(\xi - \delta_n(\xi), \xi + \delta_n(\xi)) \subset (a,b) \backslash X_n$ when $\xi \in (a,b)\backslash X_n$ such that $\lim_{n\to\infty} \tau_n = 0$ where*

$$\tau_n(x) = \sup_D |(D) \sum_{\xi \notin X_n} f(\xi)(v-u)|$$

(the supremum being taken over all δ_n-fine divisions $D = \{([u,v],\xi)\}$ of $[a,x]$ and the sum is over $([u,v],\xi)$ in D with $\xi \notin X_n$) and $\tau_n = \sup_{a \le x \le b} \tau_n(x)$.

Definition 2 *A function f is said to be LH integrable on $[a,b]$ if there is a sequence of closed subsets of $[a,b]$ with $X_n \subset X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = [a,b]$ such that f fulfills both the condition (L) and the condition (H) on $\{X_n\}$. The LH integral of f on $[a,b]$ is given by*

$$(LH)\int_a^b f(t)\,dt = \lim_{n\to\infty} (L) \int_{X_n \cap [a,b]} f(t)\,dt.$$

We take $\lim_{n\to\infty}(L)\int_{X_n\cap[a,x]} f(t)\,dt$ to be the LH primitive of f on $[a,b]$.

The LH integral is uniquely determined in view of Theorem 1 below. Obviously every Lebesgue integrable function is LH integrable there.

Theorem 1 *If f is LH integrable on $[a,b]$, then it is Henstock integrable there, and*

$$(H)\int_a^b f(x)\,dx = (LH)\int_a^b f(x)\,dx.$$

PROOF. Since f is LH integrable on $[a,b]$, there is a sequence $\{X_n\}$ of closed subsets of $[a,b]$ with $X_n \subset X_{n+1}$ for all n and $\bigcup_{n=1}^{\infty} X_n = [a,b]$ such that f fulfills both the condition (L) and the condition (H) on $\{X_n\}$. Then for every $\varepsilon > 0$ there exist an integer N and $\delta_N(\xi) > 0$ such that

$$|F(x) - (L)\int_{X_N \cap [a,x]} f(t)\,dt| < \varepsilon/3 \text{ for all } x \in [a,b]$$

and

$$\sup_D |(D)\sum_{\xi \notin X_N} f(\xi)(v-u)| < \varepsilon/3$$

where $F(x) = \lim_{n\to\infty}(L)\int_{X_n \cap [a,x]} f(t)\,dt$, the supremum being taken over all δ_N-fine divisions $D = \{([u,v],\xi)\}$ of $[a,b]$ and the sum is over $([u,v],\xi)$ in D with $\xi \notin X_n$.

Put $f_N(x) = f(x)$ when $x \in X_N$ and $f_N(x) = 0$ otherwise. Then f_N is Lebesgue and therefore Henstock integrable on $[a,b]$ with the primitive F_N. In view of Henstock lemma, for given $\varepsilon > 0$ there is a $\delta(\xi) > 0$ such that for any δ-fine partial division $D' = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$|(D')\sum\{f_N(\xi)(v-u) - (F_N(v) - F_N(u))\}| < \varepsilon/3.$$

We may assume that $\delta(\xi) \le \delta_N(\xi)$ for all $\xi \in [a,b]$. Then for any δ-fine division $D = \{([u,v],\xi\}$ of $[a,b]$ we have

$$\begin{aligned}
|(D)\sum f(\xi)(v-u) - F(b)| &\le |(D)\sum_{\xi\in X_N} f(\xi)(v-u) - F_N(b)| \\
&\quad + |(D)\sum_{\xi\notin X_N} f(\xi)(v-u) - (F(b) - F_N(b))| \\
&= |(D)\sum_{\xi\in X_N}\{f_N(\xi)(v-u) - (F_N(v) - F_N(u))\}| \\
&\quad + |(D)\sum_{\xi\notin X_N} f(\xi)(v-u) - (F(b) - F_N(b))| \\
&\le |(D)\sum_{\xi\in X_N}\{f_N(\xi)(v-u) - (F_N(v) - F_N(u))\}| \\
&\quad + |(D)\sum_{\xi\notin X_N} f(\xi)(v-u)| + |(F(b) - F_N(b))| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Hence f is Henstock integrable on $[a,b]$ and

$$(H)\int_a^b f(x)\,dx = (LH)\int_a^b f(x)\,dx.$$

□

We remark that the above theorem is still true if "f fulfills the condition (L) on $\{X_n\}$" and "$\lim_{n\to\infty}\tau_n = 0$" are replaced by "$(L)\int_{X_n} f(t)\,dt$ converges" and " $\lim_{n\to\infty}\tau_n(b) = 0''$ respectively.

On the other hand, we have the following theorem.

Theorem 2 *Let f be Henstock integrable on $[a,b]$. If f fulfills the condition (L) on $\{X_n\}$ and*

$$\lim_{n\to\infty}(L)\int_{X_n\cap[a,x]} f(t)\,dt = (H)\int_a^x f(t)\,dt \text{ uniformly on } [a,b],$$

then f also fulfills the condition (H) on $\{X_n\}$.

PROOF. For each positive integer k choose $n(k)$ so that whenever $n \ge n(k)$

$$|(L)\int_{X_n\cap[a,x]} f(t)\,dt - (H)\int_a^x f(t)\,dt| < \frac{1}{2k} \text{ for all } x\in[a,b].$$

We may assume $n(k+1) > n(k)$ for all k.

Put $f_n(x) = f(x)$ when $x \in X_n$ and $f_n(x) = 0$ otherwise. Since $\{f(x) - f_n(x)\}$ is Henstock integrable on$[a,b]$ with primitive $(H)\int_a^b (f(x) - f_n(x))\,dx$, by the Henstock Lemma for each n ($n(k) \le n < n(k+1)$) there corresponds $\delta_n(\xi) > 0$ satisfying $(\xi - \delta_n(\xi), \xi + \delta_n(\xi)) \subset (a,b)\setminus X_n$ when $\xi \in (a,b)\setminus X_n$ such that for any δ_n-fine division $D = \{([u,v],\xi)\}$ of $[a,x]$ we have

$$|(D)\sum(f(\xi) - f_n(\xi))(v-u) - (H)\int_a^x (f(t) - f_n(t))\,dt| < \frac{1}{2k}.$$

It follows that for any δ_n-fine division D of $[a,x]$ ($n(k) \le n < n(k+1)$) we have

$$|(D)\sum_{\xi\notin X_n} f(\xi)(v-u)| < \frac{1}{k}$$

i.e. $\tau_n(x) = \sup_D |(D)\sum_{\xi\notin X_n} f(\xi)(v-u)| \le \frac{1}{k}$ for each $x\in[a,b]$ and $n = n(k), n(k)+1, \ldots, n(k+1)-1$. Consequently $\tau_n \le \frac{1}{k}$ for $n = n(k), n(k)+1, \cdots, n(k+1)-1$ where τ_n and $\tau_n(x)$ are the same as above. Therefore $\lim_{n\to\infty}\tau_n = 0$. That is, f fulfills the condition (H) on $\{X_n\}$. □

To prove that every Henstock integrable function is also LH integrable, we need the following lemmas.

Lemma 3 (Cauchy extension). *If a function f defined on $[a,b]$ is LH integrable on $[a,u]$ for each $u \in [a,b)$ and*

$$\lim_{u\to b}(LH)\int_a^u f(x)\,dx = A \ \ \textit{exists}$$

then f is LH integrable to A on $[a,b]$.

PROOF. Let $a = a_1 < a_2 < \cdots$ and $a_k \to b$ as $k \to \infty$. Put $g_0(x) = f(x)$ when $x = b$ and $g_0(x) = 0$ otherwise, $g_k(x) = f(x)$ when $x \in [a_k, a_{k+1}]$ and $g_k(x) = 0$ otherwise, and $F(x) = (LH)\int_a^x f(t)\,dt$ when $x \in [a,b)$ and $F(b) = A$. It is easy to see that

$$(L)\int_a^x g_0(t)\,dt + \sum_{k=1}^{\infty}(LH)\int_a^x g_k(t)\,dt = F(x) \ \text{ uniformly on } [a,b].$$

Since g_k is LH integrable on $[a_k, a_{k+1}]$, there exists a sequence $\{X_{k,l}\}_{l\geq 1}$ of closed subsets of $[a_k, a_{k+1}]$ with $X_{k,l} \subset X_{k,l+1}$ for all l and $\bigcup_{l=1}^{\infty} X_{k,l} = [a_k, a_{k+1}]$ such that g_k fulfills both the condition (L) and the condition (H) on $\{X_{k,l}\}_{l\geq 1}$.

Now for each positive integer n there exists an integer $k(n)$ such that

$$|(L)\int_a^x g_0(t)\,dt + \sum_{k=1}^{k(n)}(LH)\int_a^x g_k(t)\,dt - F(x)| < \frac{1}{2n} \ \text{ for all } x \in [a,b].$$

Further, there exists an integer $\ell(n)$ such that

$$|(L)\int_{X_{k,l(n)}\cap[a,x]} g_k(t)\,dt - (LH)\int_a^x g_k(t)\,dt| < \frac{1}{n2^{k+1}}$$

for all $x \in [a,b]$ and for $k = 1,2,\ldots,k(n)$. We may assume $k(n+1) > k(n)$ and $l(n+1) > l(n)$ for all n. Put

$$X_0 = \{b\}, \ \ X_n = \left(\bigcup_{k=1}^{k(n)} X_{k,l(n)}\right) \cup \{X_0\} \ \ n = 1,2,\ldots.$$

It follows that

$$|(L)\int_{X_n\cap[a,x]} f(t)\,dt - F(x)|$$

$$= |(L)\int_a^x g_0(t)\,dt + \sum_{k=1}^{k(n)}(L)\int_{X_{k,l(n)}\cap[a,x]} g_k(t)\,dt - F(x)|$$
$$\leq \left|\sum_{k=1}^{k(n)}(L)\int_{X_{k,l(n)}\cap[a,x]} g_k(t)\,dt - \sum_{k=1}^{k(n)}(LH)\int_a^x g_k(t)\,dt\right|$$
$$+ |(L)\int_a^b g_0(t)\,dt + \sum_{k=1}^{k(n)}(LH)\int_a^b g_k(t)\,dt - F(x)|$$
$$< \sum_{k=1}^{k(n)} \frac{1}{n2^{k+1}} + \frac{1}{2n}$$
$$< \frac{1}{n} \text{ for all } x \in [a,b].$$

Thus f fulfills the condition (L) on $\{X_n\}$ and $\lim_{n\to\infty}(L)\int_{X_n\cap[a,x]} f(t)\,dt = F(x)$ uniformly on $[a,b]$.

On the other hand, in view of Theorem 1 and the Cauchy extension for the Henstock integral [3; p.41], f is Henstock integrable to $F(b)$ on $[a,b]$. Then in view of Theorem 2, f also fulfills the condition (H) on $\{X_n\}$. Hence f is LH integrable on $[a,b]$ and

$$(LH)\int_a^b f(t)\,dt = \lim_{n\to\infty}(L)\int_{X_n\cap[a,b]} f(t)\,dt = F(b) = A.$$

□

Lemma 4 (Harnack extension). *Let X be a closed set in $[a,b]$ and $(a,b)\setminus X$ the union of (c_k,d_k), $k=1,2,\dots$. If f is Lebesgue integrable on X and LH integrable on each $[c_k,d_k]$ with*

$$\sum_{k=1}^{\infty}\omega(F_k;[c_k,d_k]) < +\infty$$

where F_k denotes the LH primitive of f over $[c_k,d_k]$, then f is LH integrable on $[a,b]$ and

$$(LH)\int_a^b f(x)\,dx = (L)\int_X f(x)\,dx + \sum_{k=1}^{\infty}(LH)\int_{c_k}^{d_k} f(x)\,dx.$$

PROOF. Let $g_0(x) = f(x)$ when $x \in X$ and $g_0(x) = 0$ otherwise, and $g_k(x) = f(x)$ when $x \in [c_k,d_k]$ and $g_k(x) = 0$ otherwise. Since $\sum_{k=1}^{\infty}\omega(F_k;[c_k,d_k]) <$

$+\infty$, $\sum_{k=1}^{\infty}(LH)\int_a^x g_k(t)\,dt$ converges uniformly on $[a,b]$. Write $F(x) = (L)\int_a^x g_0(t)\,dt + \sum_{k=1}^{\infty}(LH)\int_a^x g_k(t)\,dt$. The rest of the proof follows in exactly the same way as that of Lemma 3. Hence f is LH integrable on $[a,b]$ and the required equality holds. □

Theorem 3 *If f is Henstock integrable on $[a,b]$, then it is LH integrable there, and*

$$(LH)\int_a^b f(x)\,dx = (H)\int_a^b f(x)\,dx.$$

PROOF. Let F be the Henstock primitive of f on $[a,b]$. We say that an interval $I \subset [a,b]$ is regular if the function f is LH integrable on I and if the function F defined on I is the LH primitive of f. Further, we say that a point $x \in [a,b]$ is regular if each sufficiently small interval $I \subset [a,b]$ containing x is regular. Let P be the set of the non-regular points of $[a,b]$. Then the set P is closed and every subinterval of $[a,b]$ which contains no points of this set is regular. In view of F being ACG^* on $[a,b]$ and Baire's category Theorem [3; p.46], f is Lebesgue integrable and therefore LH integrable on some interval in $[a,b]$ with the primitive F. In other words, the set of regular points is nonempty. We have to prove that indeed the set P is empty.

Suppose, if possible, that $P \neq \emptyset$. By Lemma 3 we see easily that every interval contiguous to P is regular and that the set P therefore has no isolated points. Again, in view of Baire's category theorem, there is a portion P_0 of P such that F is $AC^*(P_0)$. Let J_0 be the smallest closed interval containing P_0. Since the set P has no isolated points, the same is true of any portion of P, and therefore $P \cap J_0^0 \neq \emptyset$. It follows that in order to obtain a contradiction, which will justify our assertion, we need only prove that the interval J_0 is regular.

To show this, let J be any subinterval of J_0 and let Q be the set consisting of the points of the set $P \cap J$ and of the end-points of J. We denote by $\{I_n\}$ the sequence of the intervals contiguous to Q. Now the function f is Lebesgue integrable on Q and LH integrable on each interval I_n and moreover, F is the LH primitive of f on each of these intervals. Since F is also $AC^*(Q)$, the series of the oscillations of F on the intervals I_n is convergent. It follows from Lemma 4 that the function f is LH integrable on J and

$$(LH)\int_J f(x)\,dx = \sum_n F(I_n) + (L)\int_Q f(x)\,dx.$$

On the other hand, in view of Theorem 1 and the Harnack extension for the Henstock integral [3; p.41], we obtain

$$(H)\int_J f(x)\,dx = \sum_n F(I_n) + (L)\int_Q f(x)\,dx.$$

Thus $(LH)\int_J f(x)\,dx = F(J)$. Therefore, since J is any subinterval of J_0, the interval J_0 is regular and this completes the proof. □In conclusion, in view of Theorems 1 and 3, the LH integral and the Henstock integral are equivalent. The condition (L) is a necessary condition of Henstock integrability, though not sufficient as the following example from Tolstoff [4] shows.

Let K be the Cantor set and

$$I_1^{(1)} = (1/3, 2/3),\ \ I_2^{(1)} = (1/9, 2/9),\ \ I_2^{(2)} = (7/9, 8/9),$$
$$\dots$$
$$I_n^{(1)}, I_n^{(2)}, \dots, I_n^{(2^{n-1})}, \dots.$$

Write $I_n^{(k)} = (a_{nk}, c_{nk}) \cup [c_{nk}, d_{nk}] \cup (d_{nk}, b_{nk})$ and assume

$$\frac{c_{nk} - a_{nk}}{b_{nk} - a_{nk}} = \frac{b_{nk} - d_{nk}}{b_{nk} - a_{nk}} \to 0 \text{ as } n \to \infty.$$

Define $F(x) = 0$ when $x \in K$, and for each $I_n^{(k)}$ define $F(x) = 1/n$ when $c_{nk} \le x \le d_{nk}$ and linearly in $(a_{nk}, c_{nk}), (d_{nk}, b_{nk})$ so that F is a continuous function on $[0,1]$. Obviously if we take $X_1, X_2, X_3, \dots$ to be respectively K and the closures of $K \cup I_1^{(1)}, K \cup I_1^{(1)} \cup I_2^{(1)}, K \cup I_1^{(1)} \cup I_2^{(1)} \cup I_2^{(2)}, \dots$, then the condition (L) holds with $f(x) = F'(x)$ almost everywhere. But as shown in [4] the function F is not ACG^* on $[0,1]$ and therefore f is not Henstock integrable on $[0,1]$.

Finally, we state without proof the following convergence theorem. It is a generalization of the Vitali's theorem [5; VI, §3] for Henstock integral.

Theorem 4 *Let $\{f_n\}$ be a sequence of LH integrable functions on $[a,b]$. If the following conditions are satisfied:*

(i) $f_n(x) \to f(x)$ *almost everywhere in* $[a,b]$ *as* $n \to \infty$,

(ii) there is a sequence $\{X_k\}$ of closed subsets of $[a,b]$ independent of n such that $X_k \subset X_{k+1}$ for all k and $\bigcup_{k=1}^{\infty} X_k = [a,b]$, f_n is Lebesgue integrable on each X_k, the functions of the sequence $\{f_n(x)\}$ have equi-absolutely continuous integrals on X_k for each fixed k,

(iii) $(L)\int_{X_k \cap [a,x]} f_n(t)\,dt$ *converge uniformly in x and uniformly in n,*

(iv) for each k there exists a $\delta_k(\xi) > 0$ independent of n with $(\xi - \delta_k(\xi), \xi + \delta_k(\xi)) \subset (a,b) \setminus X_k$ when $\xi \in (a,b) \setminus X_k$ such that $\lim_{k\to\infty} \tau_{nk} = 0$ *uniformly in n (where*

$$\tau_{nk} = \sup_{a \le x \le b}, \tau_{nk},\ \tau_{nk}(x) \sup_D |(D) \sum_{\xi \notin X_k} f_n(\xi)(v - u)|$$

the supremum being taken over all δ_k-fine divisions $D = \{([u,v],\xi)\}$ of $[a,x]$ and the sum is over $([u,v],\xi) \in D$ with $\xi \notin X_k$), then f is LH integrable on $[a,b]$ and

$$(LH)\int_a^b f(x)\,dx = \lim_{n\to\infty}(LH)\int_a^b f_n(x)\,dx.$$

Remark 1 1) In the definition of LH integral, if "f fulfills the condition (L) on $\{X_n\}$" and "$\lim k \to \infty \tau_n = 0$" are replaced by "$f$ is Lebesgue integrable on each X_n and $(L)\int_{X_n\cap[a,x]} f(t)\,dt$ converges pointwise to a continuous function on $[a,b]$ (i.e., f fulfills the Nakanishi condition on $\{X_n\}$)" and "for each $x \in [a,b]$, $\lim_{n\to\infty}\tau_n(x) = 0''$ respectively, the integral with the weaker condition is called the WLH integral. Again, if "f fulfills the condition (L) on $\{X_n\}$"and "$\lim k \to \infty\tau_n = 0$" are replaced by "$f$ is Lebesgue integrable on each X_n, and $(L)\int_{X_n} f(t)\,dt$ converges (i.e., f fulfills the Lee Peng Yee condition on $\{X_n\}$)" and "$\lim k \to \infty\tau_n(b) = 0$" respectively, the integral with the weakest condition is called the $MWLH$ integral. From Theorem 3and the remark of Theorem 1, the LH integral the WLH integral, the $MWLH$ integral, and the Henstock integral are all equivalent.

2) In the definitions of the LH and the WLH integrals the

$$\lim_{n\to\infty}(L)\int_{X_n\cap[a,x]} f(t)\,dt$$

is in fact the primitive of f on $[a,b]$; while in the definition of the $MWLH$ integral, we only know $(L)\int_{X_n\cap[a,x]} f(t)\,dt$ converges at $x = b$.

3) For the WLH integral, Theorem 4 is still true if (iii) and (iv) are replaced respectively by (iii)$'$; for each fixed $x \in [a,b]$, $(L)\int_{X_k\cap[a,x]} f_n(t)\,dt$ converge uniformly in n and (iv)$'$; for each fixed $x \in [a,b]$, $\lim k \to \infty\tau_n(x) = 0$ uniformly in n. Again for the $MWLH$ integral Theorem 4 is valid if (iii) and (iv) are replaced by (iii)$''$; $(L)\int_{X_k} f_n(t)\,dt$ converges uniformly in n, and (iv)$''$; $\lim k \to \infty\tau_n(b) = 0$ uniformly in n.

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