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ARCWISE ALMOST CONTINUOUS FUNCTIONS

In this paper we introduce the property of the arcwise almost continuity and investigate relationships between the notions of almost continuity and arcwise almost continuity.

1. Introduction.

The notion of almost continuity introduced by Stallings [7] has been investigated in many directions, in particular to generalize the Brower fixed point theorem. In this paper we introduce the property of the arcwise almost continuity. (Note that an analogous notion of the arcwise Darboux property was introduced by Pawlak in [6].) We consider relationships between the almost continuity and the arcwise almost continuity. In particular we give an example of an arcwise almost continuous function from $\mathbf{I}^2$ into $\mathbf{I}^2$ with no fixed points (such a function is not almost continuous). Following ideas from [5] we define the notion of $(K, G)_a$ pairs of topological spaces X, Y. For such pairs (X, Y) we give a method of construction of arcwise almost continuous functions from X into Y. Moreover, we study several topological and algebraic properties of arcwise almost continuous functions analogous to that considered in [5] for almost continuous functions.

We shall use the notation introduced in [5]. In particular, symbols X, Y denote topological spaces, $\mathbb{R}$ denotes the real line and $\mathbf{I}$ denotes the unit interval. The symbol $\mathcal{A}(X, Y)$ denotes the class of all almost continuous functions (in the sense of Stallings) from X into Y. Recall that $f : X \longrightarrow Y$ is *almost continuous* iff for any open set $U \subset X \times Y$ containing f, U contains

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a continuous function $g : X \longrightarrow Y$ [7] (we consider a function and its graph to be coincident). Note that $f : X \longrightarrow Y$ is almost continuous iff it intersects every blocking set in $X \times Y$ (see [3] or [5], Remark 1.1 (p. 464)). The symbol $\mathcal{D}_P(X,Y)$ denotes the family of all functions having *Pawlak's property*, i.e. functions $f : X \longrightarrow Y$ such that $f(L)$ is connected whenever L is an arc in X [6]. Note that if X is a Hausdorff space and Y is hereditarily normal, then $\mathcal{A}(X,Y) \subset \mathcal{D}_P(X,Y)$ (see [5], Theorem 1.2 (p. 472); unfortunately, the assumption on X was omitted in that theorem.)

2. Basic properties.

A function $f : X \longrightarrow Y$ is *arcwise almost continuous* (in the sense of Stallings) iff $f|L$ is almost continuous for each arc $L \in X$. The class of all arcwise almost continuous functions from X into Y will be denoted by $\mathcal{A}_a(X,Y)$.

Note that [5], Theorem 3.4 (p. 480) implies the following characterization of the arcwise almost continuity.

Remark 2.1 *A function $f : X \longrightarrow Y$ is arcwise almost continuous iff $f \circ h$ is almost continuous for every homeomorphic injection $h : I \longrightarrow X$.*

Example 2.1 *There exists a topological space X such that*

$$\mathcal{A}(X,\mathbf{I}) \setminus \mathcal{A}_a(X,\mathbf{I}) \neq \emptyset.$$

Indeed, let τ be the topology on $X = \mathbf{I} \cup \{2\}$ defined by the following condition:

- $U \in \tau$ iff either U is open in the Euclidean topology on $\mathbf{I}$, or $U = \{2\} \cup V$, where V is an open neighbourhood of 1 in $\mathbf{I}$.

It is easy to observe that the function $f : X \longrightarrow \mathbf{I}$ defined by

$$f(x) = \begin{cases} 0 & \text{for } x = 1, \\ 1 & \text{otherwise} \end{cases}$$

is almost continuous (since each neighbourhood of f in $X \times \mathbf{I}$ contains a constant function $g \equiv 1$), but $f|\mathbf{I}$ is not almost continuous. Hence f is not arcwise almost continuous. □

However, [5], Theorem 2.1 (p. 473) yields the following result.

Theorem 2.1 *Suppose that X is a Hausdorff space. Then every almost continuous function is arcwise almost continuous.*

From [5], Lemma 2.2 (p. 474) and Lemma 2.3 (p. 475) we conclude that

Theorem 2.2 *Suppose that $A \subset \mathbb{R}$ and Y_0 is a convex subset of $\mathbb{R}^k$. Then*

$$\mathcal{A}_a(A, Y_0) = \mathcal{A}(A, Y_0).$$

Example 2.2 *Let $f : [-1,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be Lipiński's function (see [5], Example 1.7 (p. 472)). Then*

$$f \in \mathcal{A}_a([-1,1] \times \mathbb{R}, \mathbb{R}) \setminus \mathcal{A}([-1,1] \times \mathbb{R}, \mathbb{R}).$$

Indeed, let L be an arc in $[-1,1] \times \mathbb{R} \longrightarrow \mathbb{R}$. Then there exists a compact interval $J \subset \mathbb{R}$ such that $L \subset [-1,1] \times J$. By [5], Corollary 4.2,(1) (p. 487), $f|[-1,1] \times J$ is almost continuous and so is $f|L$. □

Example 2.3 *There exists $f \in \mathcal{A}_a([-1,1]^2, \mathbb{R}) \setminus \mathcal{A}([-1,1]^2, \mathbb{R})$.*

Indeed, let $A_0 = \{0\} \times [-1,1]$, $A_1 = \{(x_1, x_2) \in [-1,1]^2 : x_1 \neq 0 \text{ and } x_2 = sin(1/x)\}$ and $A = A_0 \cup A_1$. Note that A is a continuum and for each arc $L \in [-1,1]^2$ only the following cases are possible: $L \subset A_0$, $L \subset A_1$ or $card(L \setminus A) = 2^\omega$. Let $(L_\alpha)_{\alpha<2^\omega}$ be a sequence of all arcs $L \subset [-1,1]^2$ such that $L \not\subset A$. For each $\alpha < 2^\omega$ let $h_\alpha : \mathbf{I} \longrightarrow L_\alpha$ be a homeomorphism. Let $(F_\alpha)_{\alpha<2^\omega}$ be a sequence of all minimal blocking sets F in $\mathbf{I} \times \mathbb{R}$. Fix a bijection $\varphi : 2^\omega \longrightarrow 2^\omega \times 2^\omega$, $\varphi = (\varphi_1, \varphi_2)$ and set

$$K_\alpha = \left\{(x_1, x_2, x_3) \in L_{\varphi_1(\alpha)} \times \mathbb{R} : \left(h^{-1}_{\varphi_1(\alpha)}(x_1, x_2), x_3\right) \in F_{\varphi_2(\alpha)}\right\}.$$

(Note that for each $\beta < 2^\omega$, $\{K_\alpha : \varphi_1(\alpha) = \beta\}$ is a blocking family in $L_\beta \times \mathbb{R}$, see [5], p. 466.)

Now choose (inductively) a sequence $(x_\alpha, y_\alpha, z_\alpha)_{\alpha<2^\omega}$ of points such that:

(1) $(x_\alpha, y_\alpha, z_\alpha) \in K_\alpha$ for $\alpha < 2^\omega$,

(2) $(x_\alpha, y_\alpha) \notin A$ for $\alpha < 2^\omega$,

(3) $(x_\alpha, y_\alpha) \neq (x_\beta, y_\beta)$ for $\alpha \neq \beta$.

Set

$$f(x,y) = \begin{cases} i & \text{if } (x,y) \in A_i,\ i = 0,1, \\ z_\alpha & \text{if } (x,y) = (x_\alpha, y_\alpha),\ \alpha < 2^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in \mathcal{A}_a([-1,1]^2, \mathbb{R})$. Indeed, suppose that f is not arcwise almost continuous. Then $f|L_\alpha$ is not almost continuous for some $\alpha < 2^\omega$. Therefore, $f \circ h_\alpha \notin \mathcal{A}(\mathbf{I}, \mathbb{R})$ and $(f \circ h_\alpha) \cap F_\beta = \emptyset$ for some $\beta < 2^\omega$. Let $\gamma = \varphi^{-1}(\alpha, \beta)$. Then $f \cap K_\gamma = \emptyset$, contrary to $(x_\gamma, y_\gamma, z_\gamma) \in f \cap K_\gamma$.

Moreover, since $rng(f|A) = \{0, 1\}$, [5], Theorem 1.7 (p. 468), yields that $f|A$ is not almost continuous. By [5], Theorem 2.1 (p. 473), f is not almost continuous, either. □

By [5], Theorem 1.7 (p. 468) and Brown's Theorem [2] (see [5], Theorem 1.10,(4), p. 471), we conclude the following relations.

Theorem 2.3 *Assume that Y is a hereditarily normal space. Then*

(1) $\mathcal{A}_a(X,Y) \subset \mathcal{D}_P(X,Y)$,

(2) $\mathcal{B}_1(X,Y) \cap \mathcal{A}_a(X,Y) = \mathcal{B}_1(X,Y) \cap \mathcal{D}_P(X,Y)$.

Remark 2.2 *If* $f \in \mathcal{A}_a(X,Y)$ *and* $X_0 \subset X$ *then* $f|X_0 \in \mathcal{A}_a(X_0, Y)$.

Remark 2.3

(1) If $f \in \mathcal{A}_a(X,Y)$ *and* $g \in \mathcal{C}(Y,Z)$ *then* $g \circ f \in \mathcal{A}_a(X,Z)$.

(2) If Y is a Hausdorff space, $f \in \mathcal{C}(X,Y)$ *and* $g \in \mathcal{A}(Y,Z)$, *then* $g \circ f \in \mathcal{A}_a(X,Z)$.

(3) If Y is a Hausdorff space, $f : X \longrightarrow Y$ *is a continuous injection and* $g \in \mathcal{A}_a(Y,Z)$ *then* $g \circ f \in \mathcal{A}_a(X,Z)$.

PROOF. The statements (1) and (2) follow from [5], Theorems 3.3, and 3.5 (p. 480), respectively. To prove (3) note that $f|L$ is a homeomorphic injection for every arc $L \subset X$ and we can apply [5], Corollary 3.1 (p. 480). □

[5], Theorem 4.4 (p. 486), yields the following fact.

Remark 2.4 *Assume that* $f_1 \in \mathcal{A}_a(X,Y)$ *and* $f_2 \in \mathcal{C}(X,Z)$. *Then*

$$f_1 \Delta f_2 \in \mathcal{A}_a(X, Y \times Z).$$

Suppose that X and Y are topological spaces and $\mathcal{L}$ is a family of all arcs in X. We say that (X,Y) is an $(K,G)_a$ pair (*arcwise Kellum-Garret pair*) iff there exists a system $\{\mathcal{F}_L\}_{L\in\mathcal{L}}$ of families of blocking sets such that

(1) for each $L \in \mathcal{L}$, $\mathcal{F}_L$ is a blocking family in $L \times Y$ (i.e. for each $f \notin \mathcal{A}(L,Y)$ there exists a set $F \in \mathcal{F}_L$ which is blocking for f),

(2) $card(dom(F)) \geq \sum_{L\in\mathcal{L}} card(\mathcal{F}_L)$ for each $F \in \bigcup_{L\in\mathcal{L}} \mathcal{F}_L$.

The family $\mathcal{F} = \bigcup_{L \in \mathcal{L}} \mathcal{F}_L$ will be called the *blocking family* for the pair (X, Y).

Suppose that (X, Y) is an $(K, G)_a$ pair, L is an arc in X, $h : \mathbf{I} \longrightarrow L$ is a homeomorphism, $f : L \longrightarrow Y$ and $F \subset L \times Y$. It is easy to observe that F is a (minimal) blocking set for f iff $F_0 = \{(x, y) \in \mathbf{I} \times Y : (h^{-1}(x), y) \in F\}$ is a (minimal) blocking set for $f \circ h$ in $\mathbf{I} \times Y$. Thus if Y is a non-degenerate convex subset of $\mathbb{R}^k$, then $card(dom(F)) = 2^\omega$ for each blocking set F and $card(\mathcal{F}_L) = 2^\omega$ for each $L \in \mathcal{L}$ (cf [5], Proposition 1.1,(2), p. 466). Moreover, in this case we can take the family of all sets of the form $\{(x, y) \in L \times Y : (h^{-1}(x), y) \in F\}$, where F is a minimal blocking set in $\mathbf{I} \times Y$, as the family $\mathcal{F}_L$. The union $\bigcup_{L \in \mathcal{L}} \mathcal{F}_L$ of such families $\mathcal{F}_L$ we shall denote by $\mathcal{K}$ and call the *family of minimal blocking sets* for (X, Y). Note that $dom(F)$ is a connected subset of an arc L for each set $F \in \mathcal{K}$ which is blocking in $L \times Y$ (thus $dom(F)$ contains an arc). The foregoing gives the following class of examples of $(K, G)_a$ pairs.

Proposition 2.1 *Suppose that Y is a convex subset of a space $\mathbb{R}^k$ and X is a topological space with $card(\mathcal{L}) \leq 2^\omega$, where $\mathcal{L}$ denotes the class of all arcs in X. Then (X, Y) is an $(K, G)_a$ pair.*

Corollary 2.1 *If X is a second countable Hausdorff space and Y is a convex subset of $\mathbb{R}^k$ then (X, Y) is an $(K, G)_a$ pair.*

Lemma 2.1 *Suppose that (X, Y) is an $(K, G)_a$ pair, A is a subset of a Hausdorff space X and Y is a convex subset of a space $\mathbb{R}^k$. If A satisfies the following condition:*

- *(i) if L is an arc in X and $h : I \longrightarrow L$ is a homeomorphism then $h^{-1}(L \setminus A)$ is bilaterally c-dense in itself (in I),*

then

- *(ii) there exists a function $g : X \setminus A \longrightarrow Y$ such that $f \cup g$ is arcwise almost continuous for each arcwise almost continuous function $f : A \longrightarrow Y$ and $rng(g|L) = Y$ for each arc $L \not\subset A$.*

PROOF. Obviously we can assume that Y is non-degenerate. Suppose that (X, Y) is an $(K, G)_a$ pair with the family $\mathcal{K}$ of minimal blocking sets, and $\mathcal{L}$ is the family of all arcs in X. Since (X, Y) is an $(K, G)_a$ pair, either $\mathcal{L} = \emptyset$ or $card(\mathcal{L}) = 2^\omega$. Since in the first case $\mathcal{A}_a(X, Y) = Y^X$, we shall consider only the second case. Arrange all arcs $L \not\subset A$ in the sequence $(L_\alpha)_{\alpha < 2^\omega}$ and all elements of Y in the sequence $(t_\alpha)_{\alpha < 2^\omega}$. For each $\alpha < 2^\omega$ let $(F_{\alpha,\beta})_{\beta < 2^\omega}$ be a sequence of all sets $F \in \mathcal{K}$ which are blocking in $L_\alpha \times Y$ and such that $dom(F)$ is contained in the closure of no component of $int_{L_\alpha}(L_\alpha \cap A)$. By (i), $card(dom(F) \setminus A) = 2^\omega$. Fix a bijection $\varphi : 2^\omega \longrightarrow 2 \times 2^\omega \times 2^\omega$, $\varphi = (\varphi_0, \varphi_1, \varphi_2)$. For each $\alpha < 2^\omega$ choose (inductively) (x_α, y_α) such that:

(1)
$$x_\alpha \in L_{\varphi_1(\alpha)} \setminus \bigcup_J cl_{L_{\varphi_1(\alpha)}}(J),$$
where J is a component of $int_{L_{\varphi_1(\alpha)}}\left(L_{\varphi_1(\alpha)} \cap A\right)$,

(2) if $\varphi_0(\alpha) = 0$ then $y_\alpha = t_\alpha$,

(3) if $\varphi_0(\alpha) = 1$ then $(x_\alpha, y_\alpha) \in F_{\varphi_1(\alpha),\varphi_2(\alpha)}$,

(4) $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$.

Now define $g : X \setminus A \longrightarrow Y$ by
$$g(x) = \begin{cases} y_\alpha & \text{if } x = x_\alpha,\ \alpha < 2^\omega, \\ y_0 & \text{otherwise.} \end{cases}$$

Let $f : A \longrightarrow Y$ be an arbitrary arcwise almost continuous function. Let L be an arc in X and let F be a minimal blocking set in $L \times Y$. Then either $dom(F) \subset cl_L(J) \subset A$ for some component J of $int_L(L \cap A)$, or $L = L_\alpha$ and $F = F_{\alpha,\beta}$ for some $\alpha, \beta < 2^\omega$. In the first case F is blocking in $cl_L(J) \times Y$ and therefore $f \cap F \neq \emptyset$. Otherwise $(x_\gamma, y_\gamma) \in F \cap g$ for $\gamma = \varphi^{-1}(1, \alpha, \beta)$. Thus $F \cap (f \cup g) \neq \emptyset$, and consequently $(f \cup g)|L$ is almost continuous. Hence $f \cup g \in \mathcal{A}_a(X, Y)$.

Now suppose that L is an arc in X, $L \not\subset A$ and $y \in Y$. Then $y = t_\beta$ for some $\beta < 2^\omega$. Then $L = L_\alpha$ for some $\alpha < 2^\omega$ and $g(x_{\varphi(0,\alpha,\beta)}) = y$. Thus $rng(g|L) = Y$ for each arc $L \not\subset A$. □

Recall that each almost continuous function $f : X \longrightarrow X$ has a fixed point whenever X is a Hausdorff space with the fixed point property ([7], cf [5], Theorem 1.1 (p. 464)). The next example shows that the analogous property does not hold for arcwise almost continuous functions.

Example 2.4 *There exists $f \in \mathcal{A}_a(I^2, I^2)$ such that $f(x,y) \neq (x,y)$ for each $(x,y) \in I^2$ (thus f is not almost continuous).*

Indeed, let $A_0 = \{0\} \times \mathbf{I}$, $A_1 = \{(x, |sin(1/x)|) : x > 0\}$ and $A = A_0 \cup A_1$. Define $f_0 : A \longrightarrow \mathbf{I}$ by:
$$f_0(x) = \begin{cases} 1 & \text{if } x \in A_0, \\ 0 & \text{if } x \in A_1. \end{cases}$$

Obviously f_0 is arcwise almost continuous. By Lemma 2.1 (p. 512), there exists an arcwise almost continuous function $f_1 : \mathbf{I}^2 \longrightarrow \mathbf{I}$ such that $f_1|A = f_0$ and $rng(f_1|L) = \mathbf{I}$ for each arc $L \subset \mathbf{I}^2 \setminus A$. Now define $f_2 : \mathbf{I}^2 \longrightarrow I$ as follows:
$$f_2(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ |sin(1/x)| & \text{if } x > 0, \end{cases}$$

and set $f = f_1 \Delta f_2$.

We shall verify that f is arcwise almost continuous. Fix a homeomorphic injection $h : \mathbf{I} \longrightarrow \mathbf{I}^2$. Then

(1) $F = h^{-1}(A_0)$ is a closed subset of I and $(f \circ h)|F \equiv (1,0)$.

(2) If J is a component of $\mathbf{I} \setminus F$ then $(f_1 \circ h)|J$ is almost continuous and $(f_2 \circ h)|J$ is continuous. Hence $(f \circ h)|J$ is almost continuous.

(3) Suppose that $a \in F$ is the left end point of some component J of the set $\mathbf{I} \setminus F$. Then there exists a sequence $(I_n)_n$ of pairwise disjoint closed intervals in $J \setminus h^{-1}(A_1)$ such that $I_n \searrow a$ and $rng((f_2 \circ h)|I_n) \subset [0, 1/n)$ for $n \in N$. Since $rng(h|I_n)$ is an arc in $\mathbf{I}^2 \setminus A$, there exists $t_n \in I_n$ such that $(f_1 \circ h)(t_n) = 1$. Thus $(1,0) \in C^+(f \circ h, a)$. Similarly, if $a \in F$ is the right end point of some component J of $J \setminus F$ then $(1,0) \in C^-(f \circ h, a)$.

By [5], Lemma 4.1 (p. 486), we conclude that $f \circ h$ is almost continuous. Thus f is arcwise almost continuous.

Now assume that $f(x,y) = (x,y)$ for some $(x,y) \in \mathbf{I}^2$. Evidently, $x \neq 0$, so $f(x,y) = (f_1(x,y), |sin(1/x)|)$. Thus $x = f_1(x,y)$ and $y = |sin(1/x)|$ and consequently, $(x,y) \in A_1$ and $f_1(x,y) = 0$, a contradiction. □

Proposition 2.2 *Assume that (X,Y) is an $(K,G)_a$ pair, X is a Hausdorff space and $Y = \mathbb{R}^k$. Then for every normal subspace A of X the following conditions are equivalent:*

(i) each arcwise almost continuous function $f : A \longrightarrow Y$ can be extended to an arcwise almost continuous function $f^* : X \longrightarrow Y$,

(ii) each continuous function $f : A \longrightarrow Y$ can be extended to an arcwise almost continuous function $f^* : X \longrightarrow Y$,

(iii) if L is an arc in X and $h : I \longrightarrow L$ is a homeomorphism then $h^{-1}(L \setminus A)$ is bilaterally c-dense in itself (in I),

(iv) there exists a function $g : X \setminus A \longrightarrow Y$ such that $f \cup g$ is arcwise almost continuous for each arcwise almost continuous function $f : A \longrightarrow Y$ and $rng(g|L) = Y$ for each arc $L \not\subset A$,

(v) there exists a function $g : X \setminus A \longrightarrow Y$ such that $f \cup g$ is arcwise almost continuous for each arcwise almost continuous function $f : A \longrightarrow Y$.

PROOF. The implications $(v) \Rightarrow (i) \Rightarrow (ii)$ and $(iv) \Rightarrow (v)$ are obvious. $(iii) \Rightarrow (iv)$ follows from Lemma 2.1 (p. 512).

$(ii) \Rightarrow (iii)$ Assume that L is an arc in X, $h : \mathbf{I} \longrightarrow L$ is a homeomorphism and $h^{-1}(L \setminus A)$ is not bilaterally c-dense in itself (in $\mathbf{I}$). By [5], Theorem 2.3 (p. 476), there exists a continuous function $f_0 : h^{-1}(L \cap A) \longrightarrow Y$ which cannot be extended to an almost continuous function defined on whole $\mathbf{I}$. Let f be a continuous extension of $f_0 \circ h^{-1}$ on whole A. Suppose that f^* is an arcwise almost continuous extension of f on whole X. Then $f^*|L$ is almost continuous and, by [5], Theorem 3.4 (p. 480), $f^* \circ h$ is almost continuous, too. Since

$$(f^* \circ h)|h^{-1}(L \cap A) = (f \circ h)|h^{-1}(L \cap A) = f_0|h^{-1}(L \cap A),$$

$f^* \circ h$ is an arcwise almost continuous extension of f_0 on whole $\mathbf{I}$, a contradiction. □

Corollary 2.2 *Suppose that L is an arc in a Hausdorff space X, $Y = \mathbb{R}^k$ and (X, Y) is an $(K, G)_a$ pair. Then for each $f \in \mathcal{A}_a(L, Y)$ there exists $f^* \in \mathcal{A}_a(X, Y)$ such that $f^*|L = f$.*

Proposition 2.3 *Suppose that (X, Y) is an $(K, G)_a$ pair, $A \subset X$ and Y is a non-degenerate convex subset of a space $\mathbb{R}^k$. Then the following conditions are equivalent:*

(*i*) *each function $g : X \setminus A \longrightarrow Y$ can be extended to an arcwise almost continuous function $f^* : X \longrightarrow Y$,*

(*ii*) *$card(A \cap L) = 2^\omega$ for each arc L in X,*

(*iii*) *there exists $f : A \longrightarrow Y$ such that $f \cup g$ is arcwise almost continuous for every $g : X \setminus A \longrightarrow Y$.*

PROOF. $(i) \Rightarrow (ii)$ Assume that $card(A \cap L) < 2^\omega$ for some arc $L \subset X$. Fix $y_0, y_1 \in Y$ such that $y_0 \neq y_1$, $x_0 \in L \setminus A$ and define $g : X \setminus A \longrightarrow Y$ by

$$g(x) = \begin{cases} y_0 & \text{if } x = x_0, \\ y_1 & \text{otherwise.} \end{cases}$$

Suppose that $g^* \in \mathcal{A}_a(X, Y)$ is an extension of g. Then $g^*|L$ is almost continuous and $2 \leq card(rng(g^*|L)) < 2^\omega$ (so $rng(g^*|L)$ is not connected), contrary to [5], Theorem 1.7 (p. 468).

$(ii) \Rightarrow (iii)$ Arrange all arcs in X in the sequence $(L_\alpha)_{\alpha<2^\omega}$. For each $\alpha < 2^\omega$ let $(K_{\alpha,\beta})_{\beta<2^\omega}$ be a sequence of all minimal blocking sets in $L_\alpha \times Y$. Fix a bijection $\varphi : 2^\omega \longrightarrow 2^\omega \times 2^\omega$. For each $\alpha < 2^\omega$ choose $(x_\alpha, y_\alpha) \in K_{\varphi(\alpha)}$ such that $x_\alpha \in A$ and $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Set

$$f(x) = \begin{cases} y_\alpha & \text{if } x = x_\alpha,\ \alpha < 2^\omega, \\ y_0 & \text{otherwise.} \end{cases}$$

Then

$$f \cup g \in \mathcal{A}_a(X,Y)$$

for each $g : X \setminus A \longrightarrow Y$. The implication $(iii) \Rightarrow (i)$ is obvious.

□

Corollary 2.3 *For each $A \subset \mathbb{R}$ and non-degenerate convex subset Y of a space $\mathbb{R}^k$ the following conditions are equivalent:*

(*i*) *each $g : \mathbb{R} \setminus A \longrightarrow Y$ can be extended to $g^* \in \mathcal{A}_a(\mathbb{R}, Y)$,*

(*ii*) *A is c-dense in $\mathbb{R}$,*

(*iii*) *there exists $f : A \longrightarrow Y$ such that $f \cup g \in \mathcal{A}_a(\mathbb{R}, Y)$ for each $g : \mathbb{R} \setminus A \longrightarrow Y$.*

Corollary 2.4 *Let Y be a non-degenerate convex subset of a space $\mathbb{R}^k$, $M \subset X$ and (X,Y) be an $(K,G)_a$ pair. Then the following conditions are equivalent:*

(*i*) *M is a $(g, \mathcal{A}_a(X,Y))$-negligible for some arcwise almost continuous function $g : X \longrightarrow Y$ (see [1], [4], or [5], Theorem 8.3 (p. 513)),*

(*ii*) *$L \setminus M$ is c-dense in L for each arc $L \subset X$.*

3. Limits and operations.

Similarly to [5], Lemma 5.1 (p. 489) and Propositions 5.1 (p. 490), 6.1 (p. 494), 6.2 (p. 498) and 6.4 (p. 505), respectively, we can prove the following results.

Lemma 3.1 *Suppose that (X,Y) is an $(K,G)_a$ pair, $\mathcal{F}$ is an infinite blocking family for (X,Y) and $\kappa \leq \lambda = card(\mathcal{F})$. Then there exists a partition of X into κ many sets X_α $(\alpha < \kappa)$, such that $card(dom(F) \cap X_\alpha) \geq \lambda$ for each $\alpha < \kappa$ and $F \in \mathcal{F}$.*

Proposition 3.1 *Suppose that (X,Y) is an $(K,G)_a$ pair, $\mathcal{F}$ is an infinite blocking family for (X,Y) and $(\Sigma, \preceq)$ is a directed set such that $card(\mathcal{F}) \geq card(\Sigma)$. Then each function $f : X \longrightarrow Y$ is a discrete limit of a net $(f_\sigma)_{\sigma \in \Sigma}$ of arcwise almost continuous functions from X into Y.*

Proposition 3.2 *Suppose that $(Y,+)$ is a topological group, (X,Y) is an $(K,G)_a$ pair with an infinite blocking family $\mathcal{F}$ and $\mathcal{X}$ is a family of functions from X into Y such that $\kappa = card(\mathcal{X}) \leq \lambda = card(\mathcal{F})$. Then the following condition holds:*

- *there exists $g : X \longrightarrow Y$ such that $g + f \in \mathcal{A}_a(X,Y)$ for all $f \in \mathcal{X}$.*

In particular, each function f from X into Y can be expressed as a sum of two arcwise almost continuous functions from X into Y.

Proposition 3.3 *Suppose that either $F = \mathbb{R}$ or F is the field of all complex numbers, (X, F) is an $(K, G)_a$ pair with an infinite blocking family $\mathcal{F}$, V is a unitary vector space over F, $dim(V) > 1$ and " $\cdot$ " is a scalar product in V. Then each function $f : X \longrightarrow F$ can be expressed as a scalar product of two arcwise almost continuous functions $f_1, f_2 : X \longrightarrow V$.*

Proposition 3.4 *Assume that (X,Y) is an $(K,G)_a$ pair with an infinite blocking family $\mathcal{F}$ and Y is a lattice. Then*

$$\mathcal{L}(\mathcal{A}_a(X,Y)) = Y^X.$$

More precisely, each function f from X into Y can be expressed as

$$min(max(f_1, f_2), max(f_3, f_4)),$$

where f_1, f_2, f_3, f_4 are arcwise almost continuous.

Corollary 3.1 *Suppose that X is a second countable Hausdorff space, Y is a separable normed space and $f \in Y^X$. Then:*

(1) f is a discrete limit of a sequence of arcwise almost continuous functions from X into Y.

(2) f is a transfinite limit of a sequence of arcwise almost continuous functions from X into Y.

(3) f is a sum of two arcwise almost continuous functions from X into Y.

(4) $f = min(max(f_1, f_2), max(f_3, f_4))$ for some arcwise almost continuous functions f_1, f_2, f_3, f_4 from X into Y.

For topological spaces X and Y let $\mathcal{C}_a(X,Y)$ denote the class of all *arcwise continuous* functions, i.e. such functions from X into Y that $f|L$ is continuous for each arc L in X.

Theorem 3.1 *Let X be a second countable Hausdorff space. Then:*

(1) For each positive integer k,

$$\mathcal{M}_a(\mathcal{A}_a(X,\mathbb{R}^k)) = \mathcal{C}_a(X,\mathbb{R}^k).$$

(2) $\mathcal{M}_l(\mathcal{A}_a(X,\mathbb{R})) = \mathcal{C}_a(X,\mathbb{R}),$

(3) $\mathcal{M}_m(\mathcal{A}_a(X,\mathbb{R})) = \mathcal{M}(X,\mathbb{R})$.

PROOF. (1) "$\subset$" Suppose that $f \notin \mathcal{C}_a(X,Y)$. Let L be an arc in X such that $f|L$ is not continuous and let $h : I \longrightarrow L$ be a homeomorphism. Since $f \circ h \notin \mathcal{C}(I,Y)$, [5], Corollary 6.3 (p. 497), yields $f \circ h + g \notin \mathcal{A}(I,Y)$ for some $g \in \mathcal{A}(I,Y)$. By Corollary 2.2 (p. 515), there exists $g_1 \in \mathcal{A}_a(X,Y)$ such that $g_1|L = g \circ h^{-1}$. Then $(f+g_1)|L \notin \mathcal{A}(L,Y)$, so $f+g_1 \notin \mathcal{A}_a(X,Y)$ and therefore, $f \notin \mathcal{M}_a(\mathcal{A}_a(X,Y))$.

"$\supset$" Assume that $f \in \mathcal{C}_a(X,Y)$, $g \in \mathcal{A}_a(X,Y)$, L is an arc in X and $h : I \longrightarrow L$ is a homeomorphism. Then $f \circ h \in \mathcal{C}(I,Y)$, $g \circ h \in \mathcal{A}(I,Y)$ and by [5], Corollary 6.3 (p. 497), $(f+g)\circ h = (f \circ h) + (g \circ h) \in \mathcal{A}(I,Y)$. Hence $f+g$ is arcwise almost continuous.

The proofs of the equalities (2) and (3) are similar. □

4. Stationary and determining sets.

Suppose that $\mathcal{L}$ is a family of all arcs in X and $E \subset X$. For each ordinal γ let $\mathcal{L}_\gamma$ be a subfamily of $\mathcal{L}$ such that:

$$\mathcal{L}_0 = \{L \in \mathcal{L} : \ card(L \setminus E) < 2^\omega\},$$
$$\mathcal{L}_\gamma = \left\{L \in \mathcal{L} : \ card\left(L \setminus \left(E \cup \bigcup\bigcup_{\beta<\gamma} \mathcal{L}_\beta\right)\right) < 2^\omega\right\}.$$

Note that $\mathcal{L}_\beta \subset \mathcal{L}_\gamma$ for $\beta < \gamma$ and if $\mathcal{L}_\beta = \mathcal{L}_{\beta+1}$ for some ordinal β then $\mathcal{L}_\beta = \mathcal{L}_\gamma$ for all $\gamma > \beta$. Moreover, for any space X there exists α such that $\mathcal{L}_\alpha = \mathcal{L}_\gamma$ for all $\gamma > \alpha$. The least ordinal α such that $\mathcal{L}_\alpha = \mathcal{L}_{\alpha+1}$ will be denoted by $\lambda_{(E;X)}$ and called the *order of E in X*. Obviously,

$$\bigcup_\gamma \mathcal{L}_\gamma = \mathcal{L}_{\lambda_{(E;X)}}.$$

Example 4.1 *If $E = I \times [0,1)$, then $\lambda_{(E;I^2)} = 1$.*

Theorem 4.1 *Suppose that X is a second countable Hausdorff space and $E \subset X$. Then the following conditions are equivalent:*

(i) E is stationary for the class $\mathcal{A}_a(X,\mathbb{R})$,

(ii) $X \setminus \bigcup\mathcal{L} \subset E$ and $\mathcal{L} = \mathcal{L}_{\lambda_{(E;X)}}$.

PROOF. $(i) \Rightarrow (ii)$. Suppose that $x \in X \setminus (E \cup \bigcup\mathcal{L})$. Then the characteristic function $\chi_{\{x\}}$ of $\{x\}$ is arcwise almost continuous, $\chi_{\{x\}}|E \equiv 0$ and $\chi_{\{x\}} \not\equiv 0$, contrary to (i).

Now suppose that $\mathcal{M} = \mathcal{L} \setminus \bigcup_{\gamma<\lambda_{(E;X)}} \mathcal{L}_\gamma \neq \emptyset$ and

$$T = E \cup \mathcal{L}_{\lambda_{(E;X)}} \cup (X \setminus \bigcup \mathcal{L}).$$

Then for each $L \in \mathcal{M}$ we have:

(1) $card(L \setminus T) = 2^\omega$,

(2) $L \setminus T$ is "bilaterally" c-dense in itself (in L).

Let $(L_\alpha)_{\alpha<2^\omega}$ be a sequence of all $L \in \mathcal{M}$ (this sequence may not be one-to-one). For each $\alpha < 2^\omega$ let $(K_{\alpha,\beta})_{\beta<2^\omega}$ be a sequence of all minimal blocking sets in $L_\alpha \times \mathbb{R}$ such that $dom(K_{\alpha,\beta}) \not\subset T$ (thus $card(dom(K_{\alpha,\beta}) \setminus T) = 2^\omega$). Fix a bijection $\varphi : 2^\omega \longrightarrow 2^\omega \times 2^\omega$ and $z \in X \setminus T$. For each $\alpha < 2^\omega$ choose $(x_\alpha, y_\alpha) \in K_{\varphi(\alpha)}$ such that $x_\alpha \notin T \cup \{z\}$ and $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$. Set

$$f(x) = \begin{cases} 1 & \text{if } x = z, \\ y_\alpha & \text{if } x = x_\alpha,\ \alpha < 2^\omega, \\ 0 & \text{otherwise.} \end{cases}$$

We shall verify that $f|L$ is almost continuous for each arc L in X. Indeed, if $L \notin \mathcal{M}$ then $f|L \equiv 0$. If $L \in \mathcal{M}$ then $L = L_\alpha$ for some $\alpha < 2^\omega$. Now fix a minimal blocking set K in $L \times \mathbb{R}$ and consider two cases. If $dom(K) \subset T$ then since $0 \in rng(K)$, $(x, 0) \in K \cap f$ for some $x \in dom(K)$. If $dom(K) \not\subset T$, $K = K_{\alpha,\beta}$ for some $\beta < 2^\omega$. Then $(x_\gamma, y_\gamma) \in f \cap K$ for $\gamma = \varphi^{-1}(\alpha, \beta)$. Hence f is arcwise almost continuous.

Since $f|E \equiv 0$ and $f \not\equiv 0$, E is not stationary for the class $\mathcal{A}_a(X, \mathbb{R})$.

$(ii) \Rightarrow (i)$ Suppose that f is an arcwise almost continuous function and $f|E \equiv 0$. Then, using [5], Theorem 8.1 (p. 512), by transfinite induction one can easily prove that for each ordinal γ, $f|L \equiv 0$ whenever $L \in \mathcal{L}_\gamma$. Thus $f|L \equiv 0$ for every arc $L \in \mathcal{L}$ and $f|(X \setminus \bigcup \mathcal{L}) \equiv 0$, so $f \equiv 0$. Hence E is stationary for the class $\mathcal{A}_a(X, \mathbb{R})$. □

The following lemma is obvious.

Lemma 4.1 *Suppose that X, Y are topological spaces, $Y_0 \subset Y$ and Z is homeomorphic with Y_0. If Φ and Ψ are topological properties of functions such that $\Phi(X, Y_0) \subset \Phi(X, Y)$ and $\Psi(X, Y_0) \subset \Psi(X, Y)$, then*

$$R(\Phi(X,Y), \Psi(X,Y)) \subset R(\Phi(X,Z), \Psi(X,Z)).$$

In particular,

$$S(\Phi(X,Y)) \subset S(\Phi(X,Z)) \quad \textit{and} \quad D(\Phi(X,Y)) \subset D(\Phi(X,Z))$$

(see [5], p. 511, for definitions).

Corollary 4.1 *Suppose that $Y = \mathbb{R}^k$, X is a second countable Hausdorff space and $E \subset X$. Then the following conditions are equivalent:*

(*i*) *E is stationary for the class $\mathcal{A}_a(X,Y)$,*

(*ii*) *$X \setminus \bigcup \mathcal{L} \subset E$ and $\mathcal{L} = \mathcal{L}_{\lambda_{(E;X)}}$.*

P r o o f . Lemma 4.1 (p. 519) and Remarks 2.3 (p. 511), 2.4 (p. 511) imply

$$\mathcal{S}(\mathcal{A}_a(X,Y)) \subset \mathcal{S}(\mathcal{A}_a(X,\mathbb{R})).$$

Thus, by Theorem 4.1 (p. 518) we deduce the implication "$(i) \Rightarrow (ii)$".

$(ii) \Rightarrow (i)$ Suppose that E is not stationary for the class $\mathcal{A}_a(X,Y)$. Let $f \in \mathcal{A}_a(X,Y)$, $f|E \equiv 0$ and $f(x_0) = y_0 \neq 0$ for some $x_0 \in X$. Let W be the one-dimensional subspace of Y, $h : W \longrightarrow \mathbb{R}$ be a homeomorphism such that $h(0) = 0$ and let $\pi_W : Y \longrightarrow W$ be the projection of Y onto W. Then π_W is continuous, $\pi_W(y_0) = y_0$, $h(y_0) \neq 0$ and, by Remark 2.4 (p. 511), $f_1 = h \circ \pi_W \circ f \in \mathcal{A}_a(X,\mathbb{R})$. Since $f_1|E \equiv 0$, Theorem 4.1 (p. 518) implies $f_1 \equiv 0$. But $f_1(x_0) = h(y_0) \neq 0$, a contradiction. □

Theorem 4.2 *Suppose that $Y = \mathbb{R}^k$ and X is a Hausdorff space. Then the only determining set for the class $\mathcal{A}_a(X,Y)$ is whole X.*

PROOF. Obviously, X is a determining set for $\mathcal{A}_a(X,Y)$. Now we shall verify the opposite inclusion. First suppose that $Y = \mathbb{R}$. Let $\mathcal{L}$ be the family of all arcs in X. Suppose that there exists $x_0 \in X \setminus E$. There are two possible cases.

(*a*) $x_0 \notin \bigcup \mathcal{L}$. Then the characteristic function $\chi_{\{x_0\}}$ of the set $\{x_0\}$ is arcwise almost continuous, $\chi_{\{x_0\}}|E \equiv 0$ and $\chi_{\{x_0\}} \not\equiv 0$. Thus E is not determining for the class $\mathcal{A}_a(X,\mathbb{R})$.

(*b*) $x_0 \in L$ for some arc L in X. Let $h : I \longrightarrow L$ be a homeomorphism. By [5], Theorem 8.2 (p. 512), there exist $f_0, g_0 \in \mathcal{A}(I,\mathbb{R})$ such that $[f_0 = g_0] = I \setminus \{h^{-1}(x_0)\}$. By Proposition 2.2 (p. 514), there exists $t : X \setminus L \longrightarrow Y$ such that $f = t \cup (f_0 \circ h^{-1})$ and $g = t \cup (g_0 \circ h^{-1})$ are arcwise almost continuous. Moreover, $f|E = g|E$ and $f \neq g$, and therefore E is not determining for the class $\mathcal{A}_a(X,\mathbb{R})$.

Now the theorem follows by Lemma 4.1 (p. 519) for each k. □

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