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NOTES ON NONNEGATIVE CONVERGENT SERIES

The starting-point of this paper is the following well-known statement:

$$\text{If } \sum_{i=1}^{\infty} a_i \text{ is convergent, where } a_i \geq 0 \text{ for}$$

$$\text{every } i, \text{ then } \sum_{i=1}^{\infty} a_i^{\frac{i}{i+1}} \text{ is convergent, too.} \ldots (*)$$

We will investigate instead of the sequence of exponents $\left\{\frac{i}{i+1}\right\}$ another strictly increasing sequence, $\{c_i\}$, assuming $c_i > 0$ and $c_i \to 1$. First we give a necessary and sufficient condition for the validity of the analogue of $(*)$. Then - assuming that this condition is satisfied - we fix the sum of the original series and consider the supremum of the sums of the transformed series, so a function f is defined:

$$f(S) = \sup\left\{\sum_{i=1}^{\infty} a_i^{c_i} : \sum_{i=1}^{\infty} a_i = S\right\},$$

and we investigate the properties of this function further on. The next question is: when is this supremum a maximum? We will find that $f(S)$ is a maximum either for all S or for $S \leq S_0$ with some $S_0 > 0$ depending on the sequence $\{c_i\}$. We derive equations for f and f' in the maximum case ($S \leq S_0$), and infer that f is linear for $S \geq S_0$. We also prove results about the behavior of $f(S)$ near 0 and near ∞. In the last part of the paper we return to the special case: $c_i = \frac{i}{i+1}$. We give upper and lower estimates for $f(S)$ in this case.

Theorem 1 *Let $\{c_i\}$ be a strictly increasing sequence of positive numbers, $c_i \to 1$. Set $m(x) = \sum_{i=1}^{\infty} x^{\frac{c_i}{1-c_i}}$ and $L = \limsup_{i\to\infty} i^{1-c_i}$. The following four conditions are equivalent:*

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(i) If $\sum_{i=1}^{\infty} a_i$ is convergent ($a_i \geq 0$ for all i), then so is $\sum_{i=1}^{\infty} a_i^{c_i}$.

(ii) There exists a positive x_0 such that $m(x_0) < \infty$.

(iii) $L < \infty$.

(iv) There is a constant c such that $c_i > 1 - \frac{c}{\ln i} (i = 2, 3, \ldots)$.

If these conditions are satisfied, then

$$\sup \{x > 0 : m(x) < \infty\} = \frac{1}{L}.$$

PROOF. (ii) $\Rightarrow$ (i). Assume that $x_0 > 0$ and $m(x_0) < \infty$. Let $a_i \geq 0$ and $\sum_{i=1}^{\infty} a_i < \infty$. We need an upper bound for $a_i^{c_i}$. For every i we have either $a_i^{c_i} \leq \frac{1}{x_0} a_i$, or $a_i^{c_i} > \frac{1}{x_0} a_i$. In the last case $x_0 > a_i^{1-c_i}$ or $x_0^{\frac{c_i}{1-c_i}} > a_i^{c_i}$. This means that for all i, $a_i^{c_i} < \frac{1}{x_0} a_i + x_0^{\frac{c_i}{1-c_i}}$, and so

$$\sum_{i=1}^{\infty} a_i^{c_i} < \frac{1}{x_0} \sum_{i=1}^{\infty} a_i + m(x_0). \tag{1}$$

By our assumptions the right-hand side of (1) is finite and so (ii) implies (i).

(i) $\Longrightarrow$ (ii). In order to prove (i) $\Longrightarrow$ (ii) we need a lemma.

Lemma 1 *If (ii) is not true (that is, for all positive $x, m(x)$ is divergent), then for any positive S and M there exists a sequence $\{a_i\}$, $a_i \geq 0$ such that $\sum_{i=1}^{\infty} a_i = S$ and $\sum_{i=1}^{\infty} a_i^{c_i} \geq M$.*

PROOF. Observe that the inequality $x^{c_i} \geq Kx$ (K is a positive number) is valid, if $0 \leq x \leq x_i = \left(\frac{1}{K}\right)^{\frac{1}{1-c_i}}$. Then

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \left(\frac{1}{K}\right)^{\frac{1}{1-c_i}} = \frac{1}{K} \sum_{i=1}^{\infty} \left(\frac{1}{K}\right)^{\frac{c_i}{1-c_i}} = \frac{1}{K} m\left(\frac{1}{K}\right) = \infty.$$

Hence there is an $i_0 \geq 0$ such that $\sum_{i=1}^{i_0} x_i \leq S < \sum_{i=1}^{i_0+1} x_i$. Let $a_i = x_i$, if $1 \leq i \leq i_0$, $a_{i_0+1} = S - \sum_{i=1}^{i_0} x_i$, and $a_i = 0$, if $i > i_0 + 1$. Obviously $\sum_{i=1}^{\infty} a_i = S$ and $0 \leq a_i \leq x_i$ for all i, hence $a_i^{c_i} \geq K a_i$ by the choice of x_i. Therefore $\sum_{i=1}^{\infty} a_i^{c_i} \geq KS$. K may be chosen to be M/S, which proves the lemma. □

Now we prove (i) $\Longrightarrow$ (ii) of Theorem 1. Assume that (i) is true but (ii) is not. Let S and M be positive numbers. In view of Lemma 1 there are series $\{a_{ni}\}_{i=1}^{\infty}$, $a_{ni} \geq 0$ (where $n = 1, 2, \ldots$) with $\sum_{i=1}^{\infty} a_{ni} = \frac{S}{2^n}$, and

$\sum_{i=1}^{\infty} a_{ni}^{c_i} \ge nM$. Let $A_i = \sum_{n=1}^{\infty} a_{ni}$. (These sums are finite, since $a_{ni} \le \frac{S}{2^n}$.) It is easy to see that $\sum_{i=1}^{\infty} A_i = \sum_{n=1}^{\infty} \frac{S}{2^n} = S$. On the other hand $\sum_{i=1}^{\infty} A_i^{c_i}$ is divergent. Indeed, for arbitrary n obviously $A_i \ge a_{ni}$ and so $A_i^{c_i} \ge a_{ni}^{c_i}$, hence $\sum_{i=1}^{\infty} A_i^{c_i} \ge \sum_{i=1}^{\infty} a_{ni}^{c_i} \ge nM$. As nM can be arbitrarily large we have found such a convergent series that the transformed series is divergent, and this contradicts our hypothesis.

For the proof of (ii) $\Longleftrightarrow$ (iii) and the last assertion of the theorem we need two further lemmas.

Lemma 2 *Let $x > 0$ and $m(x) < \infty$. Then $xL \le 1$.*

PROOF. Let $\alpha_i = x^{\frac{1}{1-c_i}}$. Then $\sum_{i=1}^{\infty} \alpha_i = x \sum_{i=1}^{\infty} x^{\frac{c_i}{1-c_i}} = xm(x) < \infty$. So there is a j such that $\sum_{i=j+1}^{\infty} \alpha_i < \frac{1}{2}$, and a $k > j$ such that $j\alpha_k < \frac{1}{2}$. Clearly $\alpha_1 > \alpha_2 > \dots$, because $m(x) < \infty$ implies $x < 1$, and $\left\{\frac{1}{1-c_i}\right\}$ is strictly increasing. For $i > k$ we thus have $i\alpha_i = j\alpha_i + (i-j)\alpha_i < j\alpha_k + \alpha_{j+1} + \dots + \alpha_i < 1$, hence $xi^{1-c_i} = (i\alpha_i)^{1-c_i} < 1$. This proves that $xL \le 1$. □

Lemma 3 *Let $L < \infty$ and $0 < x < \frac{1}{L}$. Then $m(x) < \infty$.*

PROOF. Let $x < y < \frac{1}{L}$. Then $L < \frac{1}{y}$, so there is a j such that $i^{1-c_i} < \frac{1}{y}$ for $i > j$. Set $q = \frac{\ln x}{\ln y}$. Since $L \ge 1$ (because $i^{1-c_i} \ge 1$ for all i), we have $\ln x < \ln n\, y < 0$, therefore $q > 1$. Clearly $x = y^q$, and $y^{\frac{1}{1-c_i}} < \frac{1}{i}$ for $i > j$, so $x^{\frac{c_i}{1-c_i}} = \frac{1}{x} x^{\frac{1}{1-c_i}} = \frac{1}{x} y^{\frac{q}{1-c_i}} < \frac{1}{x} i^{-q}$ for $i > j$, which proves that $m(x) < \infty$. □

Now (ii) $\Longleftrightarrow$ (iii) is an immediate consequence of Lemma 2 and Lemma 3. The proof of (iii) $\Longleftrightarrow$ (iv) is left to the reader. The last assertion (i.e. $\sup\{x > 0 : m(x) < \infty\} = \frac{1}{L}$, if (i) - (iv) are satisfied, for example if $L < \infty$) also follows from Lemmas 2 and 3. □

We will always assume in the sequel that for the sequence $\{c_i\}$ the equivalent conditions (i) - (iv) are satisfied. For a fixed sequence $\{c_i\}$ we define $f(S) = \sup\left\{\sum_{i=1}^{\infty} a_i^{c_i} : \sum_{i=1}^{\infty} a_i = S\right\}$ $(S \ge 0)$. Observe that $f(S) < \infty$ by (1). (Obviously $f(0) = 0$.) This function f will be investigated below.

We shall say that $f(S)$ can be reached if there is a sequence $\{A_i\}$, $A_i \ge 0$ such that

$$\sum_{i=1}^{\infty} A_i = S \quad \text{and} \quad \sum_{i=1}^{\infty} A_i^{c_i} = f(S). \tag{2}$$

Theorem 2 *Let $p(x) = \sum_{i=1}^{\infty} c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}}$ (for $x > 0$) and $S > 0$. Then $f(S)$ can be reached if and only if there exists an $x > 0$ such that $p(x) = S$. If*

$f(S)$ can be reached, then there is only one sequence satisfying (2), namely $A_i = c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}}$, where $p(x) = S$.

PROOF. Assume first that $S = p(x)$. Let $g_i(y) = y^{c_i} - \frac{1}{x}y$ $(i = 1, 2, \ldots)$. The derivative of the ith function is $g_i'(y) = c_i y^{c_i - 1} - \frac{1}{x}$. From this it can be seen that in the interval $[0, \infty)$ the only maximum of g_i is at $A_i = c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}}$. By the choice of x, $\sum_{i=1}^{\infty} A_i = S$. Now we show that $\sum_{i=1}^{\infty} A_i^{c_i} = f(S)$. If the sequence $\{A_i'\}$ differs from $\{A_i\}$, but $\sum_{i=1}^{\infty} A_i' = S$, then from the maximum-property of the numbers A_i we have $A_i^{c_i} - \frac{1}{x}A_i \geq A_i'^{c_i} - \frac{1}{x}A_i'$ for every i. There is an i with $A_i \neq A_i'$, and in this case the above inequality is strict and so $\sum_{i=1}^{\infty} A_i^{c_i} - \frac{1}{x}S > \sum_{i=1}^{\infty} A_i'^{c_i} - \frac{1}{x}S$, hence $\sum_{i=1}^{\infty} A_i^{c_i} > \sum_{i=1}^{\infty} A_i'^{c_i}$. So, indeed $\{A_i\}$ is the only maximal sequence.

Assume now that $f(S)$ can be reached with a sequence $\{A_i\}$. Since $S > 0$, there is an $i > 1$ with $A_1 + A_i > 0$. Then the function $h_i(y) = y^{c_1} + (A_1 + A_i - y)^{c_i}\,|_{[0, A_1 + A_i]}$ has a maximum at A_1, since otherwise $\sum_{i=1}^{\infty} A_i^{c_i}$ could be increased with a suitable change of A_1 and A_i and without changing the sum of the original series. The derivative of $h_i(y)$ is $h_i'(y) = c_1 y^{c_1 - 1} - c_i(A_1 + A_i - y)^{c_i - 1}$. We see that $\lim_{y \to 0+0} h_i'(y) = \infty$ and $\lim_{y \to A_1 + A_i - 0} h_i'(y) = -\infty$, hence h_i has maximum neither at 0 nor at $(A_1 + A_i)$. In particular $A_1 = 0$ is impossible. So $h_i'(A_1) = 0$, hence $c_1 A_1^{c_1 - 1} = c_i A_i^{c_i - 1}$, and

$$A_i = c_i^{\frac{1}{1-c_i}} \left[\frac{1}{c_1} A_1^{1-c_1} \right]^{\frac{1}{1-c_i}}. \tag{3}$$

We have already seen that $A_1 > 0$. Consequently, $A_1 + A_i > 0$ for all i, and so (3) holds for all i, including $i = 1$. Hence if we write $x = \frac{1}{c_1} A_1^{1-c_1}$, then $S = \sum_{i=1}^{\infty} A_i = p(x)$, which proves the theorem. □

The two series defining the functions

$$m(x) = \sum_{i=1}^{\infty} x^{\frac{c_i}{1-c_i}} \text{ and } p(x) = \sum_{i=1}^{\infty} c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}}$$

of Theorems 1 and 2 are equiconvergent for $x \geq 0$. Indeed,

$$c_i^{\frac{1}{1-c_i}} = [1 - (1 - c_i)]^{\frac{1}{1-c_i}}$$

and, as $c_i \nearrow 1$, $c_i^{\frac{1}{1-c_i}} \nearrow \frac{1}{e}$. Therefore $c_1^{\frac{1}{1-c_1}} x\ m(x) \leq p(x) \leq \frac{1}{e} x\ m(x)$ proving the equiconvergence for $x \geq 0$. So if we define $H = \sup\{x > 0 : m(x) < \infty\}$, then also $H = \sup\{x > 0 : p(x) < \infty\}$. (By Theorem 1, $H = \frac{1}{L} = \frac{1}{\limsup_{i \to \infty} i^{1-c_i}}$.) Obviously $0 < H \leq 1$. For $0 \leq x < H$ $m(x)$

and $p(x)$ are convergent, and for $x > H$ they are divergent. But we have no information about the behavior of $m(H)$ and $p(H)$, they may be either divergent or convergent. These two cases are:

Case 1 – $m(H)$ and $p(H)$ are convergent,

Case 2 – $m(H)$ and $p(H)$ are divergent.

Both cases are possible. An example for Case 1 is

$$c_i = 1 - \frac{\ln 2}{1 + \ln i + \sqrt{\ln i}},$$

because then $H = \frac{1}{\lim_{i\to\infty} i^{1-c_i}} = \frac{1}{2}$, and

$$m(x) = \frac{1}{x}\sum_{i=1}^{\infty} x^{\frac{1+\ln i+\sqrt{\ln i}}{\ln 2}},$$

so $m(H) = m(\frac{1}{2}) = \frac{2}{e}\sum_{i=1}^{\infty}\frac{1}{i}e^{-\sqrt{\ln i}} < \infty$. For Case 2 one can take $c_i = \frac{i}{i+1}$ (this case will be discussed later on).

Lemma 4 *In Case 1 $f(S)$ can be reached if and only if $S \leq S_0$, where $S_0 = p(H)$. In Case 2 $f(S)$ can be reached for all S.*

PROOF. In Case 1 $p(H) = S_0$ is a finite number, and p is continuous in $[0, H]$, because here the series defining p is obviously uniformly convergent. So for $S \leq S_0$ there exists an x such that $S = p(x)$. However, for $S > S_0$ there is no such an x because p is increasing. In Case 2 $m(H)$ and $p(H)$ are divergent. The function p is continuous in $[0, H)$, since for any $0 < x_0 < H$, p is uniformly convergent in $[0, x_0]$. On the other hand, p takes arbitrarily large values. Thus in Case 2 for all $S > 0$, $p(x) = S$ with some x. Now using Theorem 2 the lemma is proved. □

We have seen (Theorem 2) that if $f(S)$ can be reached, then

$$S = p(x) = \sum_{i=1}^{\infty} c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}} \text{ and } f(S) = z(x) = \sum_{i=1}^{\infty} c_i^{\frac{c_i}{1-c_i}} x^{\frac{c_i}{1-c_i}}.$$

Lemma 5 *The series defining $p(x)$ and $z(x)$ are term by term differentiable in $(0, H)$.*

PROOF. It is easy to see that $\frac{1}{1-c_i} > 1$ for all i and $\frac{c_i}{1-c_i} > 1$ for sufficiently large i (since $c_i \to 1$). Obviously we may leave out a finite number of terms of $z(x)$, and so it suffices to prove the following statement:

If $F(x) = \sum_{i=1}^{\infty} b_i x^{a_i}$, where $b_i > 0$, $a_i \geq 1$, and F is convergent in $(0, H)$, then $F(x)$ is term by term differentiable in $(0, H)$.

By a well-known theorem it is enough to show that the series obtained by termwise differentiation of $F(x)$ is uniformly convergent in any interval $(0, x_0)$, where $x_0 < H$. Let $x_0 < H_0 < H$. Since $b_i x^{a_i}$ is a convex function for all i by $a_i \geq 1$, so for $0 < x < x_0$:

$$(b_i x^{a_i})' \leq \frac{b_i H_0^{a_i} - b_i x^{a_i}}{H_0 - x} \leq \frac{b_i H_0^{a_i}}{H_0 - x_0}.$$

As $\sum_{i=1}^{\infty} \frac{b_i H_0^{a_i}}{H_0 - x_0} = \frac{1}{H_0 - x_0} F(H_0) < \infty$, and this series is independent of x, hence the series $\sum_{i=1}^{\infty} (b_i x^{a_i})'$ is uniformly convergent in $(0, x_0)$, which proves the lemma. □

Theorem 3 *In Case 1 for $S < S_0$ and in Case 2 for all S:*

$$f'(S) = \frac{1}{p^{-1}(S)} = \frac{1}{x}, \quad \textit{and so} \quad f(S) = \int_0^s \frac{1}{p^{-1}(y)} dy.$$

PROOF. By Lemma 5

$$z'(x) = \sum_{i=1}^{\infty} \frac{1}{1 - c_i} c_i^{\frac{1}{1-c_i}} x^{\frac{c_i}{1-c_i} - 1},$$

and

$$p'(x) = \sum_{i=1}^{\infty} \frac{1}{1 - c_i} c_i^{\frac{1}{1-c_i}} x^{\frac{c_i}{1-c_i}},$$

and we see that $\frac{z'(x)}{p'(x)} = \frac{1}{x}$. As $f(S) = z\left(p^{-1}(S)\right)$, hence $f'(S) = \frac{z'(p^{-1}(S))}{p'(p^{-1}(S))}$, that is

$$f'(S) = \frac{1}{p^{-1}(S)} = \frac{1}{x}. \tag{4}$$

$\lim_{s \to 0} f(S) = 0$, therefore the improper integral $\int_0^s \frac{1}{p^{-1}(y)} dy$ is convergent, and by (4) $f(S) = \int_0^s \frac{1}{p^{-1}(y)} dy$. □

Lemma 6 *$f(S) - \frac{1}{H} S$ is an increasing function.*

PROOF. Let $0 \leq S_1 < S_2$. We want to prove that $f(S_2) - \frac{1}{H} S_2 \geq f(S_1) - \frac{1}{H} S_1$, or $\frac{f(S_2) - f(S_1)}{S_2 - S_1} \geq \frac{1}{H}$. Let $H' > H$. As it was seen, there is a maximum of the function $g_i(y) = y^{c_i} - \frac{1}{H'} y$ at $y_i = c_i^{\frac{1}{1-c_i}} H'^{\frac{1}{1-c_i}}$ and g_i is strictly increasing in

$[0, y_i]$. Consider a sequence $\{a_i\}$ for which $\sum_{i=1}^{\infty} a_i = S_1$. As $\sum_{i=1}^{\infty} y_i = p(H')$, i.e., $\sum_{i=1}^{\infty} y_i$ is divergent ($H' > H$), and $\sum_{i=1}^{\infty} a_i < \infty$, so there are infinitely many integers i so that $a_i < y_i$, and if these indices are $\{i_1, i_2, \ldots, i_k, \ldots\}$, then $\sum_{k=1}^{\infty} (y_{i_k} - a_{i_k})$ is divergent. Therefore for some $k_0 \geq 0$, $\sum_{k=1}^{k_0} (y_{i_k} - a_{i_k}) \leq S_2 - S_1 < \sum_{k=1}^{k_0+1} (y_{i_k} - a_{i_k})$. Now let $a'_{i_k} = y_{i_k}$ for $1 \leq k \leq k_0$, let $a'_{i_{k_0+1}} = S_2 - S_1 + a_{i_{k_0+1}} - \sum_{k=1}^{k_0} \left(a'_{i_k} - a_{i_k}\right)$, and put $a'_i = a_i$ for all other indices. From these definitions $\sum_{i=1}^{\infty} a'_i = S_2$. If $a'_i \neq a_i$, then $a_i \leq a'_i \leq y_i$, and hence $a'^{c_i}_i - \frac{1}{H'} a'_i \geq a_i^{c_i} - \frac{1}{H'} a_i$ for all i. We have from this $\sum_{i=1}^{\infty} a'^{c_i}_i - \frac{1}{H'} S_2 \geq \sum_{i=1}^{\infty} a_i^{c_i} - \frac{1}{H'} S_1$. So we have found for all sequences $\{a_i\}$ with $\sum_{i=1}^{\infty} a_i = S_1$ such a sequence $\{a'_i\}$. Hence a similar inequality is true for the suprema: $f(S_2) - \frac{1}{H'} S_2 \geq f(S_1) - \frac{1}{H'} S_1$. This may be written in the form $\frac{f(S_2)-f(S_1)}{S_2-S_1} \geq \frac{1}{H'}$. As this is valid for all $H' > H$, so also for H, which proves the lemma. □

Lemma 7 *$f(S)$ is a concave function.*

PROOF. We want to prove: if $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, and $S_1, S_2 \geq 0$, then $f(\alpha_1 S_1 + \alpha_2 S_2) \geq \alpha_1 f(S_1) + \alpha_2 f(S_2)$. Let $\sum_{i=1}^{\infty} a_i = S_1$ and $\sum_{i=1}^{\infty} b_i = S_2$. We define a new sequence, $\{d_i\} : d_i = \alpha_1 a_i + \alpha_2 b_i$. The function x^{c_i} is concave, so $d_i^{c_i} \geq \alpha_1 a_i^{c_i} + \alpha_2 b_i^{c_i}$ and for the sums $\sum_{i=1}^{\infty} d_i = \alpha_1 S_1 + \alpha_2 S_2$, while $\sum_{i=1}^{\infty} d_i^{c_i} \geq \alpha_1 \sum_{i=1}^{\infty} a_i^{c_i} + \alpha_2 \sum_{i=1}^{\infty} b_i^{c_i}$. Since for arbitrary $\{a_i\}$ and $\{b_i\}$ with sums S_1 and S_2, respectively, there is such a sequence $\{d_i\}$, therefore for the suprema the required inequality holds. □

Now we can describe $f(S)$ in the case when $f(S)$ can not be reached.

Theorem 4 *In Case 1 for $S \geq S_0$ $f(S)$ is linear: $f(S) = \frac{1}{H}S + f(S_0) - \frac{1}{H}S_0$. Also in Case 2 $\lim_{s\to\infty} \frac{f(S)}{S} = \frac{1}{H}$.*

PROOF. By Lemma 6 it is clear that for $S \geq S_0$, $f(S) \geq \frac{1}{H}S + f(S_0) - \frac{1}{H}S_0$. The converse inequality is a consequence of Theorem 3 and Lemma 7. Indeed, by Theorem 3 $\lim_{s\to s_0-0} f'(S) = \lim_{s\to s_0-0} \frac{1}{x} = \frac{1}{H}$, and so, because f is concave, for $S \geq S_0$, $\frac{f(S)-f(S_0)}{S-S_0} \leq \frac{1}{H}$, or $f(S) \leq \frac{1}{H}S + f(S_0) - \frac{1}{H}S_0$, which gives the first assertion. The second assertion results from L'Hôpital's rule, as in Case 2 $\lim_{s\to\infty} f'(S) = \lim_{s\to\infty} \frac{1}{x} = \frac{1}{H}$. □

Remark 1 $\lim_{s\to 0} \frac{f(S)}{S^{c_1}} = 1$. *Indeed, this statement follows from L'hôpital's rule, as $\lim_{s\to 0} \frac{f'(S)}{c_1 S^{c_1-1}} = 1$; in terms of x this limit is easily obtained using $f'(S) = \frac{1}{x}$ and taking instead of $p(x)$ the leading term of the series for $p(x)$. On the other hand obviously $f(S) > S^{c_1}$ for all $S > 0$, because for $a_1 = S$, $a_2 = a_3 = \cdots = 0$ we have $\sum_{i=1}^{\infty} a_i = S$ and $\sum_{i=1}^{\infty} a_i^{c_i} = S^{c_1}$, and by Theorem 2 $\{a_i\}$ can not be a maximal sequence.*

Now we turn to the special case of $c_i = \frac{i}{i+1}$. Then $m(x) = \sum_{i=1}^{\infty} x^{\frac{c_i}{1-c_i}} = \sum_{i=1}^{\infty} x^i$, and this is convergent for $0 \leq x < 1$. So statement (*) is contained in Theorem 1 as a special case. Obviously we have $H = 1$ here. $m(1)$ is divergent, therefore this case belongs to Case 2, so $f(S)$ can be reached for all S, and

$$S = p(x) = \sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i+1} x^{i+1}, \; f(S) = z(x) = \sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i} x^{i}.$$

By (1), we have for $f(S)$ the following upper bound, where x_0 is an arbitrary number from $(0,1)$: $f(S) \leq \frac{S}{x_0} + \sum_{i=1}^{\infty} x_0^i = \frac{S}{x_0} + \frac{x_0}{1-x_0}$. It is easy to verify that the minimum of the right-hand side, as a function of x_0, is $S + 2\sqrt{S}$ (this value is taken at $x_0 = \frac{\sqrt{S}}{1+\sqrt{S}}$). Hence $f(S) \leq S + 2\sqrt{S}$ is the best estimation obtained in this way. Now we prove a better result.

Theorem 5 *If $c_i = \frac{i}{i+1}$, then $f(S) < S + \sqrt{S}$ for all $S > 0$.*

PROOF. By Remark 1 $\lim_{s\to 0} \frac{f(S)}{\sqrt{S}} = 1$, because now $c_1 = \frac{1}{2}$. From this we have $\lim_{s\to 0} \frac{f(S)-S}{\sqrt{S}} = 1$. If we prove that $\frac{f(S)-S}{\sqrt{S}}$ is a strictly decreasing function, it will follow obviously that $\frac{f(S)-S}{\sqrt{S}} < 1$, so $f(S) < S + \sqrt{S}$ for $S > 0$. So now we show that the function $t(S) = \frac{f(S)-S}{\sqrt{S}}$ is strictly decreasing. Applying $f'(S) = \frac{1}{p^{-1}(S)}$ (Theorem 3) we obtain

$$t'(S) = \frac{1}{S}\left[\left(\frac{1}{p^{-1}(S)} - 1\right)\sqrt{S} - \frac{1}{2\sqrt{S}}(f(S) - S)\right].$$

It suffices to prove that

$$2\left(\frac{1}{p^{-1}(S)} - 1\right)S - f(S) + S < 0. \tag{5}$$

We know that $S = \sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i+1} x^{i+1}$, $f(S) = \sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i} x^{i}$, $x = p^{-1}(S)$, and so (5) can be written in the form

$$\left(\frac{2}{x} - 1\right)\sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i+1} x^{i+1} - \sum_{i=1}^{\infty} \left(\frac{i}{i+1}\right)^{i} x^{i} < 0,$$

or $\sum_{i=1}^{\infty} x^i \left[2\left(\frac{i}{i+1}\right)^{i+1} - \left(\frac{i-1}{i}\right)^i - \left(\frac{i}{i+1}\right)^i\right] < 0$. We show that every coefficient is negative for $i > 1$ (for $i = 1$ the coefficient is 0). Indeed, $2\left(\frac{i}{i+1}\right)^{i+1} -$

$(\frac{i-1}{i})^i - \left(\frac{i}{i+1}\right)^i = \left(\frac{i}{i+1}\right)^i \left[\frac{2i}{i+1} - 1\right] - (\frac{i-1}{i})^i = \frac{i-1}{i}\left[\left(\frac{i}{i+1}\right)^{i+1} - (\frac{i-1}{i})^{i-1}\right] <$ 0, because $\left(\frac{i}{i+1}\right)^{i+1} < \frac{1}{e} < (\frac{i-1}{i})^{i-1}$, and so the proof is finished. □

Theorem 5 is interesting only for small numbers S, because for large numbers we finally prove a stronger result.

Theorem 6 *If* $c_i = \frac{i}{i+1}$, *then the function* $f(S) - S - \frac{1}{e}\ln S$ *is strictly decreasing, and* $\lim_{s\to\infty}(f(S) - S - \frac{1}{e}\ln S) = \frac{1}{e} + K$, *where* $K = \sum_{i=1}^{\infty}\frac{1}{i+1}\left[(\frac{i}{i+1})^i - \frac{1}{e}\right]$.

PROOF. We have seen that $S = p(x) = \sum_{i=1}^{\infty} c_i^{\frac{1}{1-c_i}} x^{\frac{1}{1-c_i}}$. Using that $c_i^{\frac{1}{1-c_i}} < \frac{1}{e} < c_i^{\frac{c_i}{1-c_i}}$ we obtain $\frac{1}{e}\sum_{i=1}^{\infty} c_i x^{\frac{1}{1-c_i}} < S < \frac{1}{e}\sum_{i=1}^{\infty} x^{\frac{1}{1-c_i}}$. Substituting $c_i = \frac{i}{i+1}$ we have

$$\frac{1}{e}\sum_{i=1}^{\infty}\frac{i}{i+1}x^{i+1} = \frac{1}{e}\left[\frac{x}{1-x} - \ln\frac{1}{1-x}\right] < S < \frac{1}{e}\sum_{i=1}^{\infty} x^{i+1} = \frac{1}{e}\frac{x^2}{1-x} \tag{6}$$

and

$$x - \frac{\ln\frac{1}{1-x}}{\frac{1}{1-x}} < eS(1-x) < x^2. \tag{7}$$

If $S \to \infty$, then $x \to 1$ and $\frac{1}{1-x} \to \infty$. So by (7)

$$\lim_{s\to\infty} eS(1-x) = 1. \tag{8}$$

On the other hand, from (6) $S < \frac{1}{e}\frac{x^2}{1-x} < \frac{1}{e}\frac{x}{1-x}$, and so $\frac{1}{x} - 1 - \frac{1}{eS} < 0$. However, $\left[f(S) - S - \frac{1}{e}\ln S\right]' = \frac{1}{x} - 1 - \frac{1}{eS}$, and we obtain the first assertion of the theorem. If $S \to \infty$, then

$$\lim_{s\to\infty} S\left(\frac{1}{x} - 1\right) = \lim_{s\to\infty}\left(\frac{S}{x} - S\right) = \frac{1}{e} \tag{9}$$

by (8). Therefore we shall consider the difference $f(S) - \frac{S}{x}$ instead of the difference $f(S) - S$.

$$\begin{aligned} f(S) - \frac{S}{x} &= \sum_{i=1}^{\infty} x^i\left[\left(\frac{i}{i+1}\right)^i - \left(\frac{i}{i+1}\right)^{i+1}\right] = \sum_{i=1}^{\infty} x^i\left(\frac{i}{i+1}\right)^i\frac{1}{i+1} \\ &= \frac{1}{e}\sum_{i=1}^{\infty}\frac{x^i}{i+1} + \sum_{i=1}^{\infty}\frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i - \frac{1}{e}\right]x^i. \end{aligned} \tag{10}$$

Now

$$\begin{aligned}\frac{1}{e}\sum_{i=1}^{\infty}\frac{x^i}{i+1} &= \frac{1}{ex}\left[-x+\sum_{i=1}^{\infty}\frac{x^i}{i}\right] = -\frac{1}{e}+\frac{1}{ex}\ln\frac{1}{1-x} \\ &= -\frac{1}{e}+\frac{1}{e}\ln S+\frac{1}{e}\ln\frac{1}{(1-x)S}+\frac{1}{ex}(1-x)\ln\frac{1}{1-x}. \quad (11)\end{aligned}$$

We know that if $S \to \infty$, then $\frac{1}{(1-x)S} \to e$, hence $\frac{1}{e}\ln\frac{1}{(1-x)S} \to \frac{1}{e}$. On the other hand $\frac{1}{1-x} \to \infty$, and so the last summand tends to 0. Finally, we obtain from (11) $\lim_{s\to\infty}\left[\frac{1}{e}\sum_{i=1}^{\infty}\frac{x^i}{i+1} - \frac{1}{e}\ln S\right] = 0$. Applying this, (9) and (10) we get

$$\lim_{s\to\infty}\left[f(S)-\frac{1}{e}-S-\frac{1}{e}\ln S-\sum_{i=1}^{\infty}x^i\frac{1}{i+1}\left(\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right)\right]=0. \quad (12)$$

The series $\sum_{i=1}^{\infty}\frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right]$ is convergent, because

$$\begin{aligned}\frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right] &< \frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right]+\frac{1}{i}\left[\frac{1}{e}-\left(\frac{i}{i+1}\right)^{i+1}\right] \\ &= \frac{1}{ei(i+1)},\end{aligned}$$

and so

$$K=\sum_{i=1}^{\infty}\frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right] < \frac{1}{e}\sum_{i=1}^{\infty}\frac{1}{i(i+1)}=\frac{1}{e}.$$

If $S \to \infty$, then $x \to 1$ and obviously $\sum_{i=1}^{\infty}x^i\frac{1}{i+1}\left[\left(\frac{i}{i+1}\right)^i-\frac{1}{e}\right] \to K$. It follows by (12) that $\lim_{s\to\infty}\left(f(S)-S-\frac{1}{e}\ln S\right)=\frac{1}{e}+K$. □

Remark 2 *We can obtain both upper and lower estimates for $f(S)$ from Theorem 6. For example, if $S > 1$, then we have $\frac{1}{e}+K < f(S)-S-\frac{1}{e}\ln S < f(1)-1$, and by Theorem 5 $f(1) < 2$, so for $S > 1$*

$$S+\frac{1}{e}\ln S+\frac{1}{e}+K < f(S) < S+\frac{1}{e}\ln S+1.$$