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THE PACKING MEASURE AND SYMMETRIC DERIVATION BASIS MEASURE-II

In the preceding paper "The Packing Measure and Symmetric Derivation Basis Measure" [2], the author noticed that the proof of the theorem suggested a better method of calculating the packing measure on a specific set than what was given in [2].

The definition of the packing measure is:

Definition 1 *(Packing Measure) Let $h(\cdot)$ be any continuous, increasing function defined on the interval $[0,\infty)$ such that $h(0) = 0$. The premeasure of a set E is defined by $H_p(E) = \inf_{\delta\to 0}\{\sup[\sum_i h(2r_i) : B(x_i, r_i)$ is any sequence of pairwise disjoint balls in R^n with $x_i \in E$ and $r_i < \delta]\}$. Then, the packing measure is $h_p(E) = \inf\{\sum_i H_p(E_i) : E \subset \cup_i E_i\}$.*

The definition of symmetric derivation basis measure is:

Definition 2 *(Symmetric derivation basis measure). Let h and E be defined as in Definition 1. Let $\delta(\cdot)$ be any positive, real function. Then, $H_s(E) = \sup\{\sum_i h(2r_i) : B(x_i, r_i)$ is any sequence of pairwise disjoint balls in R^n with $x_i \in E$ and $r_i < \delta(x_i)\}$. The symmetric derivation basis measure is $h_s(E) = \inf\{H_s(E) : \delta(\cdot)$ is any positive, real function$\}$.*

It was shown in [1] that the packing measure and the symmetric derivation basis measure are the same on the real line. A referee has told the author that it should be stated that the results of [1] are valid in R^n.

It is not necessary to take the infimum over all positive, real functions as is stated in Definition 2 nor is it necessary to take the infimum over all Baire 3 functions as in [2]. The positive, real functions needed to calculate the packing measure on a specific set are as in the following definition:

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Definition 3 *[$\delta^*(\cdot)$ - functions] Let $\{E_\alpha\}$ be any countable collection of sets. Select an ordering of $\{E_\alpha\}$ such as $\{E_n\}_{n=1}^\infty$. Define $\delta^*(\cdot)$ on $\{E_n\}_{n=1}^\infty$ as follows:*

$$\delta^*(x) = \begin{cases} \delta_1 > 0 & , \quad x \in E_1 \\ \delta_2 > 0 & , \quad x \in E_2 \sim E_1 \\ \delta_n > 0 & , \quad x \in E_n \sim (E_1 \cup \cdots \cup E_{n-1}) \\ 1 & , \quad x \notin \cup_n E_n \end{cases}$$

Notice that $\delta^*(\cdot)$ depends not only on the countable collection of sets, but also the sequential ordering of the sets. If one changes the sequential ordering of the sets, a new $\delta^*(\cdot)$ results.

The measure that follows from the definition of the $\delta^*(\cdot)$ - functions is:

Definition 4 *[$\delta^*(\cdot)$ - measure] Let $h(\cdot)$ and E be defined as in Definition 1. Let $\delta^*(\cdot)$ be any $\delta^*(\cdot)$ - function whose associated sets $\{E_n\}_{n=1}^\infty$ cover E (e.g. $E \subset \cup_n E_n$). Then, $H_s{}^*(E) = \sup\{\sum_i h(2r_i) : B(x_i, r_i)$ is any sequence of pairwise disjoint balls in R^n with $x_i \in E$, $x_i \in E_n \sim (E_1 \cup \cdots \cup E_{n-1})$ for some n, and $r_i < \delta_n\}$. The $\delta^*(\cdot)$ - measure is $h_s{}^*(E) = \inf\{H_s{}^*(E) : \delta^*(\cdot)$ is any $\delta^*(\cdot)$ - function whose associated sets $\{E_n\}_{n=1}^\infty$ cover E (e.g. $E \subset \cup_n E_n$)$\}$.*

The proof of the following theorem is very similar to the theorem in [2].

Theorem 1 *For any set E and function $h(\cdot)$ defined as above, $h_s{}^*(E) = h_s(E) = h_p(E)$.*

PROOF. It is clear that a $\delta^*(\cdot)$ - function is a positive, real function. Since $h_s(E)$ is the infimum of $H_s(E)$ over all positive, real $\delta(\cdot)$ functions, $h_s(E) \leq h_s{}^*(E)$. If $h_s(E) = \infty$, then $h_s{}^*(E) = \infty$ and $h_s(E) = h_s{}^*(E)$. So, assume that $h_s(E)$ is finite. Since the symmetric derivation basis measure is the packing measure [1], $h_s(E) = \inf\{\sum_i H_p(E_i) : E \subset \cup_i E_i\}$. Let $\varepsilon > 0$ be given. Then there exists a sequence of sets $\{E_i\}$ such that $\sum_i H_p(E_i) < h_s(E) + \varepsilon$. For each i, choose $\delta_i > 0$ such that $\sum_j h(2r_{i,j}) < H_p\ (E_i) + \varepsilon/2^i$ for all pairwise disjoint sequence of balls $\{B(x_{i,j}, r_{i,j})\}_{j=1}^\infty$ with $x_{i,j} \in E_i$ and $r_{i,j} < \delta_i$. Define $\delta^*(\cdot)$ on $\cup_i E_i$ inductively by $\delta^*(x) = \delta_1$, on E_1, $\delta^*(x) = \delta_2$ on $E_2 \sim E_1$, and $\delta^*(x) = \delta_n$ on $E_n \sim (E_1 \cup \cdots \cup E_{n-1})$ for any natural number n. Define $\delta^*(x) = 1$ on the complement of $\cup_i E_i$. Since $E \subset \cup_i E_i$, for any sequence of disjoint balls $\{B(x_k, r_k)\}$ with $x_k \in E$ and $r_k < \delta^*(x_k)$, $\sum_k h(2r_k) < \sum_i H_p(E_i) + \varepsilon < h_s(E) + 2\varepsilon$. Hence $H_s{}^*(E) < h_s(E) + 2\varepsilon$ and $h_s{}^*(E) < h_s(E) + 2\varepsilon$. Therefore, $h_s{}^*(E) \leq h_s(E)$ and $h_s{}^*(E)$ is equal to the packing measure.

Example 1 Let E be the union of the Cantor Set and the points $x_1 = (1/3)+(1/3)(1/2)$, $x_2 = (1/3)+(1/3)(1/4)$, $x_3 = (1/3)+(1/3)(1/8), \ldots, x_n = (1/3) + (1/3)(1/2^n), \ldots$ in the interval $(1/3, 2/3)$. Call $I_1 = \{x_1, x_2, \ldots\}$. Then, I_2^1 consists of the points $x_1^1 = (1/9) + (1/9)(1/2)$, $x_2^1 = (1/9) + (1/9)(1/4), \ldots, x_n^1 = (1/9) + (1/9)(1/2^n), \ldots$ where I_2^1 is contained in the interval $(1/9, 2/9)$. I_2^2 is defined similarly for the interval $(7/9, 8/9)$. This pattern is repeated for all contiguous intervals to the Cantor Set. If the packing measure is calculated according to the original definition on set E, then the sum

$$P^\alpha(C) + \sum_{n=1}^{\infty} P^\alpha(\{x_n\}) + \sum_{i=1}^{2}\sum_{n=1}^{\infty} P^\alpha(\{x_n^i\}) + \cdots + \sum_{i=1}^{n}\sum_{n=1}^{\infty} P^\alpha(\{x_n^i\}) + \cdots = P^\alpha(C) = 2.$$

If the method of this paper is used, the first packing would be

$$B(0,1/2),\ B(1,1/2)$$

. The second δ's used for a packing would be

$$B(0,1/6),\ B(1/3,1/6),\ B(2/3,1/6) \text{ and } B(1,1/6).$$

In the third packing different δ's are used. The packing is

$$B(0,1/18),\ B(1/9,1/18),\ B(2/9,1/18),\ B(1/3,1/18),\ B(2/3,1/18)$$
$$B(7/9,1/18),\ B(8/9,1/18),\ B(1,1/18) \text{ and } B(1/2,1/10^3).$$

The sum becomes $2^3(1/3^2)^\alpha + (2/10^3)^\alpha$. At the next stage, using the same method, the sum becomes $2^4(1/3^3)^\alpha + 3(2/10^4)^\alpha + 2(2/10^4 3)^\alpha$. Finally, at the sixth stage, the sum becomes $2^6(1/3^5)^\alpha + 6(2/10^6)^\alpha + 2 \cdot 4(2/3^2 10^6)^\alpha + 2^2 3(2/3^4 10^6)^\alpha + 2^3(2/3^6 10^6)^\alpha$. So, the general sum is less than or equal to

$$\begin{aligned}
& 2^n(1/3^{n-1})^\alpha + \sum_{i=1}^{n}(n+1-i)(2/3^{i-1}10^n)^\alpha \\
= \ & 2^n(1/3^{n-1})^{(\log 2/\log 3)} + \sum_{i=1}^{n} 2^\alpha(n+1-i)/(3^{i-1}10^n)^\alpha \\
= \ & 2^n(1/2^{n-1}) + 2^\alpha \sum_{i=1}^{n}(1/2^{i-1})(n+1-i)(10^n)^\alpha \\
= \ & 2 + (2^\alpha/10^{n\alpha}) \cdot \sum_{i=1}^{n}(n+1-i)/2^{i-1} \\
\leq \ & 2 + (2/10^n)^\alpha(n/(1/2))2 + 2^\alpha(2n/10^{n\alpha}).
\end{aligned}$$

Since $(2n/10^{n\alpha}) \to 0$ as $n \to \infty$, the packing measure is 2 using Theorem 1.

References

[1] Sandra Meinershagen, *The Symmetric Derivation Basis Measure and the Packing Measure*, Proc. Amer. Math Soc., **103** (1988), No 3, 813–814.

[2] Sandra Meinershagen, *The Packing Measure and Symmetric Derivation Basis Measure*, Real Analysis Exchange, **17**, No. 1 (1991–92).