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# STRONGLY BALANCED SELECTIONS *

## 1. Introduction

The notion of path derivates and derivative is introduced in [1] and the notion of selective derivates and derivative is introduced in [2]. It is proved (Theorem 3.4 in [1]) that for a system of paths $E$ that is bilateral and satisfies the internal intersection condition there is a selection $s$ such that every $E$-differentiable function $f$ is $s$-differentiable and $sf'(x) = f'_E(x)$ for every $x$. A partial answer to whether every selective derivative $sf'$ can be realized as a path derivative $f'_E$ is provided in [5] (p. 113) and is stated as a theorem here.

In general, selective derivatives do not have the property that a selectively differentiable monotone function is differentiable. This is pointed out by O'Malley in [2]. Hence conditions need to be imposed on selections so that this property holds. In this paper, a condition is found.

Let $R$ denote the real line. We state some definitions from [1], [2], and [3] here.

For $x \in \mathbb{R}$, a *path leading to* $x$ is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and $x$ is a point of accumulation of $E_x$. A *system of paths* is a collection $E = \{E_x\colon x \in \mathbb{R}\}$ such that each $E_x$ is a path leading to $x$. For such a system $E$ the *E-derivates* $\underline{f}'_E(x)$, $\bar{f}'_E(x)$ and the *E-derivative* $f'_E(x)$ of a function $f$ at a point $x$ is just respectively the usual derivates and derivative at $x$ relative to the set $E_x$.

A system of paths $E = \{E_x\colon x \in \mathbb{R}\}$ is said to be *bilateral* at $x$ if $x$ is a bilateral point of accumulation of $E_x$, and *nonporous* at $x$ if $E_x$ has porosity zero at $x$. If $E$ has any of these properties at each $x$, then we say that $E$ has that property. $E$ is said to satisfy the *internal intersection condition* (*IIC*) if there exists a positive function $\delta$ such that $E_x \cap E_y \cap (x, y) \neq \emptyset$ whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$.

*This paper is dedicated to the memory of Prof. Tsing-houa Teng.

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For convenience, we let $[x,y]$ denote the interval having $x$ and $y$ as endpoints regardless of $x < y$ or $x > y$. A *selection* $s$ is an interval function defined on the class of all nondegenerate closed intervals $[x,y]$ in $\mathbb{R}$ such that $s[x,y]$ is a point in $(x,y)$. It is said to be *balanced* if there exist two functions $\alpha$ and $\delta$ on $\mathbb{R}$ such that $0 < \alpha(x) < 1$ and $\delta(x) > 0$ for each $x \in \mathbb{R}$ and $s[x,y] \in (x,y)_{\alpha(x)}$ if $0 < |x-y| < \delta(x)$, where $(x,y)_{\alpha(x)}$ is the interval

$$\left(\frac{x+y}{2} - \alpha(x)\frac{|x-y|}{2}, \quad \frac{x+y}{2} + \alpha(x)\frac{|x-y|}{2}\right).$$

(This notation will be used in the sequel.)

Let $s$ be a selection. The selective derivates and derivative with respect to $s$, or simply the $s$-derivates and $s$-derivatives, of a function $f$ at $x$ are, respectively

$$\underline{f}'(x) = \liminf_{y\to x} \frac{f(s[x,y]) - f(x)}{s[x,y]-x}$$

$$s\bar{f}'(x) = \limsup_{y\to x} \frac{f(s[x,y]) - f(x)}{s[x,y]-x},$$

and $sf'(x) =$ the limit of the same quotient if it exists.

Finally, $f$ is said to be *E-differentiable* or *s-differentiable* at $x$ if the corresponding derivative exists and is finite. When the phrase "at $x$" is omitted, we mean that this is true at each $x$.

## 2. Results

First, we state the partial answer to the open question mentioned in the introduction.

**Theorem 1** *If $s$ is a balanced selection with associated functions $\alpha$ and $\delta$, then $E = \{E_x\colon x \in \mathbb{R}\}$ with $E_x = \{s[x,y]\colon 0 < |x-y| < \delta(x)\} \cup \{x\}$ is a system of paths such that $\underline{f}'_E(x) = s\underline{f}'(x)$ and $\bar{f}'_E(x) = s\bar{f}'(x)$ for every function $f$ and every $x \in \mathbb{R}$.*

It should be noted that this system of paths satisfies an intersection condition stronger then the *IIC*. We state it as follows:

**Definition 1** *A system of paths $E = \{E_x\colon x \in \mathbb{R}\}$ is said to satisfy the* ***strong internal intersection condition*** *(SIIC) if there exist functions $\alpha$ and $\delta$ on $\mathbb{R}$ such that $0 < \alpha(x) < 1$, $\delta(x) > 0$ for each $x \in \mathbb{R}$ and $E_x \cap E_y \cap (x,y)_{\delta(x,y)} \neq \emptyset$ whenever $0 < |x-y| < \min\{\delta(x), \delta(y)\}$, where $\hat{\alpha}(x,y) = \min\{\alpha(x), \alpha(y)\}$.*

**Theorem 2** *Let $E$ be a nonporous system of paths that satisfies the SIIC. Then there exists a balanced selection $s$ such that every $E$-differentiable function $f$ is $s$-differentiable and $sf'(x) = f'_E(x)$ for all $x \in \mathbb{R}$.*

**Proof.** Let $\alpha$ and $\delta_0$ be the functions associated with the *SIIC* of $E$. Since, for each $x \in \mathbb{R}$, $E_x$ has porosity zero at $x$, there exists $\delta_1(x) > 0$ such that $E_x \cap (x,y)_{\alpha(x)} \neq \emptyset$ whenever $0 < |x-y| < \delta_1(x)$. Let $\delta = \min\{\delta_0, \delta_1\}$. Then, with the functions $\alpha$ and $\delta$, the selection $s$ defined below is easily seen to be balanced.

(i) If $0 < |x-y| < \min\{\delta(x), \delta(y)\}$, take $s[x,y]$ any point in $E_x \cap E_y \cap (x,y)_{\dot{\alpha}(x,y)}$.

(ii) if $0 < |x-y| < \delta(x)$ but $\geq \delta(y)$, take $s[x,y]$ any point in $E_x \cap (x,y)_{\alpha(x)}$.

(iii) if $|x-y| \geq \max\{\delta(x), \delta(y)\}$, take $s[x,y]$ any point in $(x,y)$.

The other part of the conclusion follows easily.

Since a nonporous system is bilateral, the hypothesis in Theorem 2 above is stronger than that in Theorem 3.4 of [1] and clearly the conclusion here is also stronger. Theorem 2 may lead us to ponder if the system in Theorem 1 is nonporous. The answer is negative.

**Example 1** For $x < y$, we define $s[x,y] = \frac{1}{2}(x+y)$ if $x \neq 0$ or $y \geq 1$. If $x = 0 < y < 1$, there exists a unique integer $n$ such that $(2/3)^{n+1} < y \leq (2/3)^n$ and we define $s[x,y] = (2/3)^{n+2}$. Let $\delta(x) = 1$ and $\alpha(x) = 1/3$ for each $x \in \mathbb{R}$. We see easily that $s$ is balanced. However,

$$E_0 = \left\{ s[0,y]:\ 0 < |y| < 1\} \cup \{0\} = \left(-\frac{1}{2}, 0\right] \cup \{(2/3)^n:\ n = 2,3,\ldots \right\}$$

is not nonporous at 0 since the porosity of $E_0$ at $x = 0$ from the right is

$$\limsup_{r \to 0+} \frac{\ell(0,r,E_0)}{r} = \frac{1}{3} > 0,$$

where $\ell(0,r,E_0)$ is the length of the largest open interval contained in $(0,r) - E_0$.

**Definition 2** *A selection $s$ is said to be strongly balanced if there exist two sequences of functions $\{\alpha_n\}$, $\{\delta_n\}$ on $\mathbb{R}$ and a dense subset $Q$ of $\mathbb{R}$ such that $0 < \alpha_n(x) < 1$, $\delta_n(x) > 0$ for each $x \in \mathbb{R}$, and each $n$, both $\{\alpha_n(x)\}$ and $\{\delta_n(x)\}$ decrease to zero for each $x \in \mathbb{R}$, and if $0 < |x-y| < \delta_n(x)$, then $s[x,y] \in (x,y)_{\alpha_n^*(x)}$, where* $\begin{aligned} \alpha_n^*(x) &= \alpha_n(x) \quad \text{if } y \in Q, \\ &= \alpha_1(x) \quad \text{if } y \notin Q. \end{aligned}$

We can show that if $s$ is a strongly balanced selection, then the system of paths $E = \{E_x\colon x \in \mathbb{R}\}$ with $E_x = \{s[x,y]\colon 0 < |x-y| < \delta_1(x)\} \cup \{x\}$ is nonporous. Before showing this, we present the following.

**Theorem 3** *Let $f$ be an approximately differentiable function and let $f'_{ap}$ denote its approximate derivative. Then there exists a strongly balanced selection $s$ such that $sf'(x) = f'_{ap}(x)$ for each $x \in \mathbb{R}$.*

**Proof.** Let $Q$ be the set of $x$ at which $f$ is differentiable. Then $Q$ is dense in $\mathbb{R}$. Also, for each $x \in \mathbb{R}$, there is a measurable set $A_x$ such that $A_x$ has density 1 at $x$ and

$$f'_{ap}(x) = \lim_{\substack{y \to x \\ y \in A_x}} \frac{f(y) - f(x)}{y - x}.$$

Let $\mu$ denote the Lebesgue measure. Then for each positive integer $n$, there is a $\delta_n(x) > 0$ such that

$$\mu(A_x \cap I) > \frac{2n+1}{2n+2}\mu(I)$$

whenever $x \in I$ and $\mu(I) < \delta_n(x)$. $\delta_n(x)$ can be chosen such that $\delta_{n+1}(x) \leq \delta_n(x)$ and $\lim_{n\to\infty} \delta_n(x) = 0$ for each $x$.

Let $|x-y| > 0$ be given. We define $J_k = (x,y)_{1/(k+1)}$ for $k = 1, 2, \ldots$. It is routine to check that

$$\begin{array}{lcll} \mu(A_y \cap J_m) & > & \frac{1}{m+1}\frac{|x-y|}{2} & \text{if } |x-y| < \delta_m(y), \\ \mu(A_x \cap J_n) & > & \frac{1}{n+1}\frac{|x-y|}{2} & \text{if } |x-y| < \delta_n(x), \end{array} \quad \text{and}$$

$$\mu(A_x \cap A_y \cap J_1) > 0 \quad \text{if} \quad |x-y| < \min\{(\delta_1(x), \delta_1(y)\}.$$

If $|x-y| < \delta_1(x)$, then there exists a largest integer $n$ such that $|x-y| < \delta_n(x)$. In the sequel, when we write $|x-y| \lhd \delta_n(x)$, we mean that, $n$ is the largest one, that is, if we also have $|x-y| < \delta_k(x)$, then $k \leq n$. $s[x,y]$ is chosen as follows:

(i) If $|x-y| \geq \max\{\delta_1(x), \delta_1(y)\}, s[x,y] \in (x,y)$.

(ii) If $\delta_1(x) \leq |x-y| \lhd \delta_m(y)$, or $|x-y| \lhd \min\{\delta_n(x), \delta_m(y)\}$ (i.e., $|x-y| \lhd \delta_n(x)$ and $|x-y| \lhd \delta_m(y)$) and $x \in Q$, $y \notin Q$, then $s[x,y] \in A_y \cap J_m$.

(iii) If $\delta_1(y) \leq |x-y| \lhd \delta_n(x)$, or $|x-y| \lhd \min\{\delta_n(x), \delta_m(y)\}$ and $x \notin Q$, $y \in Q$, then $s[x,y] \in A_x \cap J_n$.

(iv) If $|x-y| \lhd \min\{\delta_n(x), \delta_m(y)\}$ and $x \notin Q$, $y \notin Q$, then $s[x,y] \in A_x \cap A_y \cap J_1$.

(v) If $|x-y| \lhd \min\{\delta_n(x), \delta_m(y)\}$ and $x \in Q$, $y \in Q$, then $s[x,y] = \frac{1}{2}(x+y)$.

Let $\alpha_n(x) = 1/(n+1)$ for each $x \in \mathbb{R}$. Then it can be checked that $s$ is strongly balanced. Moreover, if $|x-y| < \delta_1(x)$ and $x \notin Q$, then $s[x,y] \in A_x$ and hence $sf'(x) = f'_{ap}(x)$ when $x \notin Q$. If $x \in Q$, since $f$ is differentiable at $x$, we also have $sf'(x) = f'_{ap}(x)$. The proof is completed.

**Theorem 4** *Let $s$ be a strongly balanced selection and $f$ be a monotone function on $\mathbb{R}$. Then $s\underline{f}'(x) = \underline{f}'(x)$ and $s\bar{f}'(x) = \bar{f}(x)$ for each $x \in \mathbb{R}$.*

**Proof.** Let $\{\alpha_n\}$, $\{\delta_n\}$ and $Q$ be associated with $s$ as in Definition 2. Let $E_x = \{s[x,y]\colon 0 < |x-y| < \delta_1(x)\} \cup \{x\}$ for each $x \in \mathbb{R}$. Firstly, we show that the system $E = \{E_x\colon x \in \mathbb{R}\}$ is nonporous. Suppose the contrary. That is, we assume that for some $x \in \mathbb{R}$, $E_x$ is porous at $x$, say from the right. Then there exists a sequence of positive numbers $\{h_k\}$ decreasing to zero such that, for some $\theta \in (0,1)$,

$$(*)\qquad E_x \cap \bigcup_{k=1}^{\infty} (x+\theta h_k, x+h_k) = \emptyset.$$

For each $k$, let $y_k = x + (1+\theta)h_k$. Then $y_k \in \left(x + \frac{1+3\theta}{2}h_k,\ x + \frac{3+\theta}{2}h_k\right)$ and we can pick $z_k \in Q \cap \left(x + \frac{1+3\theta}{2}h_k,\ x + \frac{3+\theta}{2}h_k\right)$. Let $n_0$ be an integer such that $\alpha_{n_0}(x) < (1-\theta)/(3+\theta)$. This is possible since $a_n(x) \searrow 0$ and $(1-\theta)/(3+\theta) > 0$. Also, there exists $k_0$ such that $h_k < \frac{2}{3+\theta}\delta_{n_0}(x)$ if $k \ge k_0$. It follows that $0 < |x - z_k| < \delta_{n_0}(x)$ if $k \ge k_0$. Since $s$ is strongly balanced, $z_k \in Q$ and $0 < |x - z_k| < \delta_{n_0}(x)$, we have

$$s[x,z_k] \in (x,z_k)_{\alpha_{n_0}(x)} =$$

$$\left(\frac{x+z_k}{2} - \alpha_{n_0}(x)\frac{|x-z_k|}{2}, \frac{x+z_k}{2} + \alpha_{n_0}(x)\frac{|x-z_k|}{2}\right).$$

Also,

$$\frac{x+z_k}{2} - \alpha_{n_0}(x)\frac{|x-z_k|}{2} > \frac{1}{2}\left(x + x + \frac{1+3\theta}{2}h_k\right) - \frac{1-\theta}{3+\theta}\frac{1}{2}\frac{3+\theta}{2}h_k = x + \theta h_k$$

$$\frac{x+z_k}{2} + \alpha_{n_0}(x)\frac{|x-z_k|}{2} < \frac{1}{2}\left(x + x + \frac{3+\theta}{2}h_k\right) + \frac{1-\theta}{3+\theta}\frac{1}{2}\frac{3+\theta}{2}h_k = x + h_k.$$

Hence $s[x,z_k] \in (x+\theta h_k, x+h_k)$. However, for $k \ge k_0$, $s[x,z_k] \in E_x$. This is a contradiction to $(*)$. Therefore $E$ is nonporous. Since $f$ is monotone, by Theorem 4.4 of [1], we have $\underline{f}'_E(x) = \underline{f}'(x)$ and $\bar{f}'_E(x) = \bar{f}'(x)$ for each $x \in \mathbb{R}$. Thus Theorem 4 follows from this and Theorem 1.

**Remark 1** *In [4] O'Malley shows that $sf'$ has the Denjoy-Clarkson property if $f$ is s-differentiable. Hence, if $s$ is balanced, then $sf'$ has the Zahorski's $\mathcal{M}_2$ property. Now, if $s$ is strongly balanced, we have shown that the corresponding system of paths $E$ is nonporous and satisfies the SIIC and hence by Theorems 6.6.1 and 6.11 in [1] and our Theorem 1, $sf'$ has the Zahorski's $\mathcal{M}_3$ property.*

## References

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