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ON LEBESGUE INTEGRABILITY OF MCSHANE INTEGRABLE FUNCTIONS

Abstract

In this note we provide another direct proof of the Lebesgue integrability of McShane integrable functions

It is well known that McShane and Lebesgue integrals coincide (see [2], [3], [4], [5]), but it is not easy to find direct proofs of the Lebesgue integrability of McShane integrable functions, and in fact we only know Kubota's proof (see [2]). In this note we provide another direct proof of this result.

First let us recall the definition of McShane integral and some known facts about it which are needed in our proof. We will refer to [2] as an easy reference.

Definition 1 *Let δ be a positive function defined on the real interval $[a,b]$. A tagged interval $(s,[c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point s in $[a,b]$ (we do not assume $s \in [c,d]$). The tagged interval $(s,[c,d])$ is subordinate to δ if*

$$[c,d] \subset (s-\delta(s), s+\delta(s))$$

Let $\mathcal{P} = \{(s_i,[c_i,d_i]) : 1 \le i \le N\}$ be a finite collection of non-overlapping subintervals of $[a,b]$. If $[a,b] = \bigcup_{i=1}^{N}[c_i,d_i]$, then $\mathcal{P}$ is called a tagged partition of $[a,b]$ (the points s_i are the taggs of $\mathcal{P}$ and the $[c_i,d_i]$ are the intervals of $\mathcal{P}$). We say that $\mathcal{P}$ is subordinate to δ if $(s_i,[c_i,d_i])$ is subordinate to δ for each i. If $f:[a,b] \longrightarrow \mathbb{R}$ is a function, we denote

$$f(\mathcal{P}) = \sum_{i=1}^{N}(d_i - c_i)f(s_i)$$

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Definition 2 *([4]) The function $f : [a,b] \longrightarrow \mathbb{R}$ is McShane integrable on $[a,b]$ if there exists $\alpha \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that*

$$|f(\mathcal{P}) - \alpha| < \varepsilon$$

whenever $\mathcal{P}$ is a tagged partition on $[a,b]$ subordinate to δ. We will denote

$$\alpha = (M)\int_a^b f$$

Proposition 1 *(Henstock's lemma [2, Proposition 3]) Let f be a McShane integrable function on $[a,b]$ and let δ be a positive function on $[a,b]$ such that*

$$|f(\mathcal{P}) - (M)\int_a^b f| < \varepsilon$$

whenever $\mathcal{P}$ is a tagged partition of $[a,b]$ subordinate to δ. Let $\{(s_i,[c_i,d_i]) : 1 \leq i \leq N\}$ one of such tagged partitions and let J be a subset of $\{1,\ldots,N\}$, then we have

$$|\sum_{j\in J}((d_j - c_j)f(s_j) - (M)\int_{c_j}^{d_j} f)| < \varepsilon$$

and

$$\sum_{j\in J}|(d_j - c_j)f(s_j) - (M)\int_{c_j}^{d_j} f| < 2\varepsilon.$$

Proposition 2 *([2, Proposition 6]) If f is a McShane integrable function on $[a,b]$, then its indefinite integral F is continuous, differentiable almost everywhere, and $F'(x) = f(x)$ almost everywhere.*

We can now give the announced proof.

Theorem 1 *(McShane) If $f : [a,b] \longrightarrow \mathbb{R}$ is McShane integrable then it is Lebesgue integrable.*

PROOF. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a McShane integrable function, let F be the indefinite integral of f, let δ be a positive function on $[a,b]$ such that

$$|f(\mathcal{P}) - (M)\int_a^b f| < 1$$

whenever $\mathcal{P}$ is a tagged partition on $[a,b]$ subordinate to δ, and let $\{(s_i,[c_i,d_i]) : 1 \leq i \leq N\}$ one of such tagged partitions. By Proposition 2, $F' = f$ almost everywhere, and theref ore, by a classical result on Lebesgue theory (see for instance [6, Theorem 8.19]), to prove that f is Lebesgue integrable on $[a,b]$ it suffices to show that F is of bounded variation on $[a,b]$, or, equivalently, on all the $[c_i,d_i]$. Fix $i_0 \in \{1,\dots,N\}$, and let $c_{i_0} = x_1 < x_2 < \cdots < x_n = d_{i_0}$. Since $\{(s_{i_0},[x_i,x_{i+1}]) : 1 \le i \le n-1\}$ are tagged intervals subordinate to δ, we deduce from Proposition 1:

$$\sum_{i=1}^{n-1} |(x_{i+1}-x_i)f(s_{i_0}) - (F(x_{i+1})-F(x_i))| \quad = \quad \sum_{i=1}^{n-1} |(x_{i+1}-x_i)f(s_{i_0}) - (M)\int_{x_i}^{x_{i+1}} f| \quad < \quad 2.$$

Therefore,

$$\sum_{i=1}^{n-1} |F(x_{i+1}) - F(x_i)| < \sum_{i=1}^{n-1} |F(x_{i+1}) - F(x_i) - (x_{i+1}-x_i)f(s_{i_0})| + \sum_{i=1}^{n-1} |f(s_{i_0})|(x_{i+1}-x_i) < 2 + |f(s_{i_0})|(d_{i_0}-c_{i_0}),$$

hence F is of bounded variation on $[c_{i_0}, d_{i_0}]$. □

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