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A NOTE ON OPEN-INTERVAL MEASURES

Abstract

We apply a type I metric outer measure construction to give a further new proof for the existence of open-invariant measures on a compact metric spaces.

1. A type I outer measure construction

A paving $\mathcal{P}$ on a set X is a class of subsets of X which includes at least the empty set. A mapping $\tau : \mathcal{P} \longrightarrow [0,1]$ is said to be a pre-measure provided that $\tau(\emptyset) = 0$. The outer measure defined as follows

$$\mu_I^\tau(E) = \inf\left\{\sum_{n=1}^{\infty} \tau(C_n);\ E \subseteq \bigcup_{n=1}^{\infty} C_n,\ C_n \in \mathcal{P}\right\}$$

is called an outer measure of type I [4]. Note that if E is the empty set then E is covered by $\emptyset \in \mathcal{P}$ and the outer measure of E is zero. Suppose now that (X, d) is a metric space and if $d(C)$ denotes the diameter of the set C, then

$$\mu_{II}^\tau(E) = \sup_{\varepsilon>0} \inf\left\{\sum_{n=1}^{\infty} \tau(C_n);\ E \subseteq \bigcup_{n=1}^{\infty} C_n,\ d(C_n) \leq \varepsilon,\ C_n \in \mathcal{P}\right\}$$

is said to be an outer measure of type II [4]. The outer measure μ_{II}^τ is always a metric outer measure since it satisfies

$$dist(E,F) > 0 \text{ implies } \mu_{II}^\tau(E \cup F) = \mu_{II}^\tau(E) + \mu_{II}^\tau(F)$$

where $dist(E,F) = \inf\{d(x,y);\ x \in E,\ y \in F\}$. Furthermore we call a paving $\mathcal{P}$ finite union and finite intersection stable iff finite unions and finite intersection of elements from $\mathcal{P}$ are in $\mathcal{P}$. Since for a metric outer measure all closed sets become measurable, it is an interesting question under which

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conditions μ_I^τ also becomes a metric outer measure. The next theorem proved in [1] gives sufficient conditions. Note that a paving $\mathcal{P}$ is said to be metrically separating iff for set A and B with $dist(A, B) > 0$ there are two elements C_A and C_B of the paving such that $\mathcal{P}$ containing A respectively B and satisfy $dist(C_A, C_B) > 0$.

Theorem 1.1 *If (X, d) is a metric space, τ a pre-measure which is defined on a metrically separating paving $\mathcal{P}$ which is stable with respect to finite unions and finite intersections then provided τ is monotone and supadditive on $\mathcal{P}$, that means*

$$C_1 \subseteq C_2 \text{ implies } \tau(C_1) \leq \tau(C_2) \text{ for } C_1, C_2 \in \mathcal{P} \text{ and}$$
$$\tau(C_1 \cup C_2) \geq \tau(C_1) + \tau(C_2) \text{ if } C_1, C_2 \in \mathcal{P} \text{ and } dist(C_1, C_2) > 0,$$

the outer measure μ_I^τ is a metric outer measure.

Moreover we have obtained [1] that

Theorem 1.2 *Under the conditions of Theorem 1.1 type I and type II outer measures are equal.*

2. A new proof for open-invariant measures

The fact that the space $M(X)$ of probability measures on a compact metric space (X, d) is weakly compact gives rise to a second new proof of a result which is due to J. Mycielski that a (compact) metric space has at least one open-invariant measure. A probability measure is said to be open-invariant iff open isometric sets get equal measure. Mycielski [3] has proved this for general metric spaces, but the proof uses Banach limits. A first new proof was given in [2] where we have defined inductively a pre-measure. In the following we will use as in [2] also the point packing number of a set E which is

$$M(E, q) = \max\{k \in N; \exists\, x_1, \ldots, x_k \in E \text{ such that } d(x_i, x_j) > q,\ i \neq j\}.$$

Let $E_n \subseteq X$ be a finite set such that $M(E_n, \frac{1}{n}) = M(X, \frac{1}{n}) = card(E_n)$. E_n is said to be an $\frac{1}{n}$-net. Finally define probability measures

$$\mu_n = \sum_{x \in E_n} \frac{1}{M(X, \frac{1}{n})} \varepsilon_x,$$

where ε_x denotes the dirac measure at x. The sequence (μ_n) has a weakly convergent subsequence which converges to some $\mu \in M(X)$. For simplicity we assume that (μ_n) is this sequence. We choose as a pre-measure for a type I construction

$$\tau(E) = \liminf_{n\to\infty} \frac{M(E, \frac{1}{n})}{M(X, \frac{1}{n})}.$$

If ν denotes now the type I outer measure arising from the above τ which is defined on the paving of all open sets we can prove that

Theorem 2.1 *The measure ν is an open-invariant probability measure.*

Proof. Since for $0 < q < dist(E, F)$ we obtain that

$$M(E \cup F, q) = M(E, q) + M(F, q)$$

and the pre-measure τ becomes supadditive. Clearly, τ is monotone in the sense that for open sets G and H such $G \subseteq H$ $\tau(G) \leq \tau(H)$ is satisfied. Further, if G and H are open isometric sets then $\tau(G) = \tau(H)$ and thus $\nu(G) = \nu(H)$. As $\nu(X) \leq 1$ it remains to verify that the opposite inequality holds. If (G_m) is any open cover of X we have

$$\tau(G_m) \geq \liminf_{n\to\infty} \mu_n(G_m)$$

since $M(G_m, \frac{1}{n})$ may be larger than the cardinality of the part of the $\frac{1}{n}$-net E_n which is contained in G_m. The weak convergence $\mu \to \mu$ is equivalent to

$$\liminf_{n\to\infty} \mu_n(G) \geq \mu(G)$$

for all open G, $\tau(G_m) \geq \mu(G_m)$ and thus $\nu(X) \geq 1$ if we sum up. □

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