INROADS

Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 448–452

Jacek Hejduk, Institute of Mathematics Łódź University, Ul. Banacha 22, 90-238 Łódź, Poland, email: jachej@plunlO51.bitnet

NON-BAIRE SETS IN CATEGORY BASES

Around 1975 John C.Morgan II introduced a theory of category bases. Its main feature is to present measure and category and some other properties of point set classification in a common framework. The aim of this paper is to give conditions on a category base under which each abundant set will contain a non-Baire set. It is a generalization of a theorem of Harasishvili (cf. th 1.3 in [2]) concerning the existence of sets without the Baire property in the topological spaces.

Let us recall some basic definitions and concepts of the theory of category bases.

A category base on a set X is a pair $(X, \mathcal{S})$ such that X is a set and $\mathcal{S}$ is a family of subsets of X, the nonempty subsets called regions, satisfying the following axioms

(1) $\bigcup \mathcal{S} = X$

(2) Let A be a region and $\mathcal{D}$ a non-empty family of disjont regions of cardinality less than the cardinality of $\mathcal{S}$.

Then

(i) if $A \cap (\bigcup \mathcal{D})$ contains a region, then there is a region $B \in \mathcal{D}$ such that $A \cap B$ contains a region

(ii) if $A \cap (\bigcup \mathcal{D})$ contains no region, then there is a region $B \subset A$ which is disjoint from $\bigcup \mathcal{D}$.

Standard examples of category bases include topologies and sets of positive measure with respect to a σ-finite measure. We shall say that a set $C \subset X$ is

Key Words: measure, category, Baire sets, base of an ideal, abundant set, meager set

Mathematical Reviews subject classification: Primary 04A03, 03E20 Secondary 28A05, 26A03

Received by the editors January 2, 1992

singular if,for every region A,there exists a region $B \subset A$ such that $B \cap C = \emptyset$. A set $M \subset X$ is meager if M is countable union of singular sets. A set Y which is not meager is called an abundant set. A set Z is abundant everywhere in a region A if, for each region $B \subset A$, $B \cap Z$ is abundant. The class of meager sets for a category base (X, S) will be denoted by $\mathcal{M}(S)$. In the case that the category base (X, S) is a topology, the class of all Baire sets is identical with the family of sets with the Baire property,and the class of meager sets is identical with the family of all sets of the first category. If the category base is the family of measurable sets of positive measure with respect to a σ-finite complete measure, then $\mathcal{B}(S)$ is the family of measurable sets and $\mathcal{M}(S)$ is the family of all sets of measure zero.

By a base of an ideal of sets $\mathcal{P}$ we shall understand a subfamily $\mathcal{P}'$ such that each member of $\mathcal{P}$ is contained in some member of $\mathcal{P}$.

Now we formulate the principal theorem of this paper.

Theorem 1 *(cf. 3.1 in [2].) Let (X, S) be a category base on an infinite set X such that the following conditions are satisfied*

1^0 $\mathcal{M}_0 \subset \mathcal{M}(S)$, *where* $\mathcal{M}_0 = \{A \subset X : card\ A < card\ X\}$

2^0 *there exists a base of the σ-ideal $\mathcal{M}(S)$ of cardinality not greater than card X.*

Then the following conditions are equivalent

(i) C is meager

(ii) each subset of C is Baire set.

The proof of this theorem will be based on the following result of Harasishvili(see th. 5.2 in [1]).

Theorem 2 (Harasishvili) . *Let X be an infinite set and $\phi_1 \subset 2^X$ such that*

1^0 $card\ \phi_1 \leq card\ X$

2^0 $\forall Z \in \phi_1\ (card\ Z = card\ X)$

Then there exists $\phi_2 \subset 2^X$ such that

a) $card\ \phi_2 > card\ X$

b) $\forall Z_1, Z_2 \in \phi_2\ (Z_1 \neq Z_2 \Rightarrow card(Z_1 \cap Z_2) < card\ X)$

c) $\forall Y \in \phi_1 \forall Z \in \phi_2\ (card(Z \cap Y) = card\ X)$.

We also utilize the followig lemma.

Lemma 1 *If $(X, \mathcal{S})$ is a category base and $\{A_\alpha : \alpha < \lambda\}$, where $\lambda \leq card\ \mathcal{S}$ is a family of essentialy disjont Baire abundant sets, then there exists a family of disjont regions $\{B_\alpha : \alpha < \lambda\}$ such that for any $\alpha < \lambda$ every set A_α is abundant everywhere in B_α.*

The proof of this lemma is similar to the proof of theorem 1.5 in [3].

Proof of the theorem. Only the implication (ii)$\Rightarrow$(i) needs a proof. We will present the proof in two steps. At first, let C be a region. Without loss of generality let us assume that $C = X$, because in the case of an arbitrary region C, we could consider a category base $(C, \mathcal{S}_C)$, where $\mathcal{S}_C = \{B \in \mathcal{S} : B \subset C\}$ for which (see th. 1.11 in [3]) $\mathcal{M}(\mathcal{S}_C) = \mathcal{M}(\mathcal{S}) \cap C$ and $\mathcal{B}(\mathcal{S}_C) = \mathcal{B}(\mathcal{S}) \cap C$. It is clear that the assumption 1^0 and 2^0 are satisfied for the category base $(C, \mathcal{S}_C)$. Let us assume that each subset of X is a Baire set and $X \notin \mathcal{M}(\mathcal{S})$. Let $\mathcal{B}^*$ denote any base of $\mathcal{M}(\mathcal{S})$ of cardinality not greater than card X. Putting

$$\phi_1 = \{Y \subset X : X - Y \in \mathcal{B}^*\}$$

we have that

$$card\ \phi_1 \leq card\ \mathcal{B}^* \leq card\ X$$

and, for each set $Y \in \phi_1$, $card\ Y = card\ X$ because otherwise, $X \in \mathcal{M}(\mathcal{S})$. Hence by Theorem H, there exists a family ϕ_2 fulfilling (a)-(c). By condition (b) and assumption 1^0 it is clear that for two distinct sets

$$Z_1, Z_2 \in \phi_2, Z_1 \cap Z_2 \in \mathcal{M}(\mathcal{S}).$$

Moreover each set $Z \in \phi_2$ is abundant. Indeed, if $Z \in \mathcal{M}(\mathcal{S})$, then there would exist a set $B \in \mathcal{B}^*$ containing Z such that $X - B \in \phi_1$. Hence by condition (c) we would obtain that $card\ (Z \cap (X - B)) = card\ X$, contraty to the fact that $Z \cap (X - B)) = \emptyset$ and that X is infinite. We prove that there exist a member of the family ϕ_2, which is not a Baire set. Now, at first, we shall conclude that ***card*** $\mathcal{S} \geq$ ***card*** ϕ_2. Namely, for each set $Z \in \phi_2$, there exists a region V such that Z is abundant everywhere in V. It is easy to see, by theorem 1.3 in [3], that for any two distinct sets $Z_1, Z_2 \in \phi_2$ the regions V_1, V_2 are also distinct. Let $\phi_2 = \{A_t\}_{t<\lambda}$. According to the lemma there exists a family of disjont regions of cardinality greater than *card* X. This contradiction completes the proof of the first step.

In the second step we assume that C is an arbitrary subset of X and each subset of C is a Baire set. Let us suppose that C is not a meager set. By the fundamental theorem (see chapter 1 in [3]) there exists a region B such that C is abudant everywhere in B. Hence B is abundant. Since C is a Baire set, we have, by theorem 1.2 in [3], that $B - C$ is a meager set. Moreover, for any subset $B' \subset B$ the equality $B' = B' \cap C \cup B' \cap (B - C)$ implies that B' is a Baire set. By the first step of our proof, B is meager. This is contrary to the fact that B is abundant.

In the case that the category base is the family of sets of positive measure over the real line, $\mathbb{R}$, or the natural topology over the real line, we can easily obtain the existence of a non measurable set or a set without the Baire property (cf. p.40 in [2]). The theorem depends on the axiom of choice. In the proof of the theorem the axiom of choice was clearly used.

Proposition 1 *If a category base $(X, \mathcal{S})$ is such that card $X \geq$ (card $\mathcal{S})^{\omega_0}$, then the property c.c.c. implies the existence of a base of $\mathcal{M}(\mathcal{S})$ of cardinality not greater than card X.*

Proof. Under the assumption of c.c.c. each meager set is contained in a set belonging to the family $\mathcal{K}_{\delta\sigma}$, where $\mathcal{K} = \{A \subset X : A = X - B, B \in \mathcal{S}\}$ (see th.1.5 in [1]). It means that the family $\mathcal{K}_{\delta\sigma}$ is a base of the family $\mathcal{M}(\mathcal{S})$ such that *card* $(\mathcal{K}_{\delta\sigma}) =$ (*card* $\mathcal{S})^{\omega_0} \leq$ *card* X.

There are no connections between the c.c.c. property for an arbitrary category base $(X, \mathcal{S})$ and the possesion of a base of cardinality not greater than *card* X by the σ -ideal of meager sets.

Example 1 *Let $2^{\omega_0} = 2^{\omega_1} = \omega_2$. There exists a category base $(X, \mathcal{S})$ such that the condition c.c.c. is not satisfied, but the family of meager sets $\mathcal{M}(\mathcal{S})$ possesses a base of cardinality not greater than card X and, moreover, $\mathcal{M}_0(\mathcal{S}) \subset \mathcal{M}(\mathcal{S})$.*

We give an example of a topological space $(\mathbb{R}, \mathcal{T})$ which has the desired properties. Let $\mathcal{T}_0$ denote the natural topology on the real line, $\mathbb{R}$. Now, we will define a topology on $\mathbb{R}$ putting $\mathcal{T}_1 = \{A \subset \mathbb{R} : A = U - X;$ where $U \in \mathcal{T}_0$ and $X \subset \mathbb{R}$, *card* $X \leq \omega_1\}$. Let us consider the family $((\mathbb{R}, \mathcal{T}_1)_i)_{i<\omega_1}$ and let $(\mathbb{R}, \mathcal{T}_2)$ denote the topological sum of the family $((\mathbb{R}, \mathcal{T}_1)_i)_{i<\omega_1}$. Futher, let $(\mathbb{R}, \mathcal{T})$ be the topological space where $\mathcal{T}$ is the topology induced by the bijection from $(\mathbb{R}, \mathcal{T}_2)$. It is clear that any set $A \subset \mathbb{R}$ such that *card* $A < 2^{\omega_0}$ is the first category. consequently $\mathcal{M}_0 \subset \mathcal{M}(\mathcal{S})$. We see that the condition c.c.c. is not satisfied. It is obvious that the family of all sets of the first category in the topological space $(\mathbb{R}, \mathcal{T}_1)$ possesses the base consisting of all sets of the form $F \cup G$ where F is $\mathcal{F}_\sigma$, set of the first category in the natural topology and G is such that $card\ G = \omega_1$. From the assumption that $2^{\omega_1} = \omega_2$ we conclude that the family of all sets of the first category in the space $(\mathbb{R}, \mathcal{T}_2)$ and consequently in the space $(\mathbb{R}, \mathcal{T})$, possesses a base of cardinality not grater than ω_2.

Example 2 *(cf. example 3.1 in [1]). Let α be the first measurable cardinal number and let X be any set of cardinality α and μ any propability measure on the family of all subsets of X such that $\mu(\{x\}) = 0$ for each $x \in X$. The category base of all sets of positive μ measure is clearly a point meager category base with the c.c.c. property and, from Theorem 1, we conclude that there does not exist a base of the family of all meager sets of cardinality not greater than α.*

Acknowledgement. I would like to thank the referees for their many valuable suggestions.

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