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THE g-INTEGRAL IS NOT ROTATION INVARIANT

1. Introduction

In [1] A. Novikov and W.F. Pfeffer verified that the *restricted gage integral*, (abbreviated as *g^*-integral*), is invariant with respect to lipeomorphic changes of coordinates. W.F. Pfeffer in a letter, [3], asked the author to try to find an example showing that the *(unrestricted) gage integral*, the *g-integral*, is not invariant even with respect to rotations. This paper contains this example.

Our result illustrates that although the definition of the g–integral is simpler, the slightly more sophisticated g^*–integral has more attractive properties. The example presented in this paper also illustrates, see Remark 2, that the family of g^*-integrable functions is a proper subset of the family of g-integrable functions (see also the Remark after Definition 8.4 in [2]).

2. Preliminaries

Put $N = \{1, 2, ...\}$. By Z and $\mathbb{R}$ we denote the integers and the real numbers. Given a set $A \subset \mathbb{R}^2$ we denote by $|A|$ the Lebesgue measure of A. The open ball of radius r centered at $x \in \mathbb{R}^2$ is denoted by $B(x, r)$. (In this paper we use the Euclidean metric, some papers use different but equivalent metric in $\mathbb{R}^2$. The integrals defined via any of these metrics is the same.) A two-dimensional interval is a set of the form $[a_1, b_1] \times [a_2, b_2]$ where $a_1 < b_1$, and $a_2 < b_2$. The regularity of an interval is the number

$$\frac{\min\{b_1 - a_1, b_2 - a_2\}}{\max\{b_1 - a_1, b_2 - a_2\}}.$$

If $1 \geq \eta > 0$ then an interval is η-regular if its regularity is not less than η. Denote by T the rotation of $\mathbb{R}^2$ by $+\frac{\pi}{4}$. This rotation maps the x-axis onto

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If $1 \geq \eta > 0$ then an interval is η-regular if its regularity is not less than η. Denote by T the rotation of $\mathbb{R}^2$ by $+\frac{\pi}{4}$. This rotation maps the x-axis onto the line $y = x$ and the y-axis onto $y = -x$. A T-interval is the T-image of an interval, that is, a closed rectangle with sides parallel to the lines $y = x$, or $y = -x$. The regularity of a T-interval A equals the regularity of the interval $T^{-1}(A)$.

A figure and a T-figure is a finite union of intervals and T-intervals respectively. The perimeter of a figure, or a T-figure C is denoted by $||C||$. (Since the boundary of a figure consists of finitely many line segments there is no doubt about the definition of the perimeter of figures.) A collection $P = \{(A_i, x_i) : i = 1, ..., p\}$ is a subpartition of the figure C if the *intervals* $A_i \subset C$ are non-overlapping and $x_i \in A_i$. The subpartition P is a partition when $\cup_{i=1}^{p} A_i = C$. A subpartition $\{(A_i, x_i) : i = 1, ..., p\}$ is η-regular if all intervals A_i, are η-regular for $i = 1, ..., p$.

A set is *thin* if it is of σ-finite one-dimensional Hausdorff measure. A non-negative function δ on a set $E \subset \mathbb{R}^2$ is called a gage on E whenever the set $\{x \in E : \delta(x) = 0\}$ is thin. Given a gage function δ on a figure C and a subpartition $P = \{(A_i, x_i) : i = 1, ..., p\}$ of C we say that P is δ-fine when $A_i \subset B(x_i, \delta(x_i))$.

For the standard definition of the gage integral we refer to Definition 6.1 of [2]. In this paper we do not use explicitly this definition. On the other hand in the next few paragraphs we summarize the properties of the g-integral we need in this paper. All these results and definitions are from [2].

Assume that C is a given figure. The function F defined on all subfigures of C is called an *additive function* in C if

$$F(C) = \sum_{i=1}^{p} F(C_i)$$

holds for all systems $\{C_i : i = 1, ..., p\}$ consisting of non-overlapping subfigures of C. Additive functions of intervals, T-figures and T-intervals are defined in the obvious way. It is also clear that any additive function of intervals can be extended to be an additive function (of figures). An additive function in C is *continuous* if for any $\epsilon > 0$ there exists an $\eta > 0$ such that $|F(B)| < \epsilon$ holds for any subfigure $B \subset C$ satisfying $||B|| < 1/\epsilon$ and $|B| < \eta$.

Recall Proposition 6.4 of [2].

Proposition 1 *If C is a figure and f is g-integrable in C, then f is g-integrable on each figure $B \subset C$, and the map $F : B \mapsto (g) \int_B f$ is an additive continuous function in A.*

We also need the Henstock Lemma for g-integrable functions and the necessary condition for g-integrability which can be obtained from its conclusion

(Lemma 6.5 of [2], see also the remark after Proposition 2.3 in [1]).

Proposition 2 *Let C be a figure and f a function defined on C. Then f is g-integrable in C if and only if there is an additive continuous function F in C such that for every $\epsilon > 0$ there exists a gage δ in C for which*

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| < \epsilon$$

holds for each δ-fine ϵ-regular partition $\{(A_i, x_i) : i = 1, ..., p\}$ of C.

For a while it was confusing for the author that the above Theorem is true for both g-integrable and g^*-integrable functions with almost the same wording. The basic difference is that in the case of g-integrable functions partitions consist of subintervals, while partitions in the theory of g^*-integrable functions consist of subfigures.

We say that an additive function F in a figure C is *derivable* at $x \in C$ if there exists a finite limit $\lim F(Q_n)/|Q_n|$ for each sequence $\{Q_n\}$ of subintervals of C containing x, satisfying $\lim_{n\to\infty} \operatorname{diam}(Q_n) = 0$ and for which there exists an $\eta > 0$ such that the regularity of all Q_n's exceeds η. When all these limits exist they have the same value, denoted by $F'(x)$.

We also state Theorem 6.6 of [2].

Proposition 3 *Let f be a g-integrable function in a figure C, and let $F(B) = (g)\int_B f$ for each figure $B \subset C$. Then for almost all $x \in C$ the function F is derivable at x and $F'(x) = f(x)$. In particular, the function f is measurable.*

Recall also that the g-integral is a generalization of the Lebesgue integral and a function is Lebesgue integrable if and only if both f and $|f|$ are g-integrable, see Theorem 6.7 of [2].

By $\int_A f$ we denote the Lebesgue integral of f on A. For the g-integral we shall use the symbol $(g)\int_A f$.

3. Main Result

Theorem 1 *The g-integral is not invariant with respect to rotations.*

PROOF. We construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is g-integrable on any subinterval of $\mathbb{R}^2$, but there are subintervals of $\mathbb{R}^2$ on which $f \circ T$ is not g–integrable.

Before turning to the details of our construction we outline its basic ideas. We define f as the infinite sum of certain auxiliary functions f_n. The functions

f_n are rather simple. These functions have non-zero values only on certain narrow stripes which are parallel to the line $y = x$. We denote by H_n^+ the union of those half-stripes where the function f_n has a non-zero positive value denoted by t_n. By H_n^- we denote the union of the other half-stripes where f_n equals $-t_n$. The choice of the constants t_n requires a sort of a compromise to make $f = \sum f_n$, g-integrable, but $f \circ T$ non-g-integrable. This goal can be achieved since the non-zero values of each f_n almost completely cancel out on each interval. On the other hand a T-interval can contain long linear parts of a half-stripe in H_n^+ without intersecting its neighboring other half-stripe in H_n^-. Hence it is possible that on relatively large subsets of a T-interval the $+t_n$ values of f_n are not canceled by some $-t_n$ values.

We turn now to the details of the proof. To define the sets H_n^+ and H_n^- we need some auxiliary sets H'_n chosen so that $H'_n \supset \cup_{k=1}^n (H_k^+ \cap H_k^-)$ and points in $\mathbb{R}^2 \setminus H'_n$ are sufficiently far away from the sets $H_k^+ \cup H_k^-$ $(k = 1, \ldots, n)$. We start the definition of the sets H_n^+, H_n^-, and H'_n by the definition of some important constants.

For $n = 1, 2, \ldots$ put

$$d_n = \frac{1}{2^{4n^2}}, \quad h_n = \frac{1}{2^{3n+1}2^{4n^2}}, \quad \text{and} \quad c_n = 2^{2n}.$$

It is easy to verify the following properties of the above constants.

$$\sum_{n=1}^{\infty} h_n < +\infty, \qquad \sum_{n=1}^{\infty} d_n < +\infty, \qquad \sum_{n=1}^{\infty} h_n d_n < +\infty, \tag{1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{c_n - 1} < +\infty, \qquad \sum_{n=1}^{\infty} \frac{d_n}{(c_n - 1)^2 h_n} < +\infty. \tag{2}$$

For $j \in Z$ put

$$a_{n,j} = j \cdot d_n - h_n, \qquad b_{n,j} = j \cdot d_n + h_n,$$

and

$$a'_{n,j} = j \cdot d_n - c_n h_n - h_{n+1}, \qquad b'_{n,j} = j \cdot d_n + c_n h_n + h_{n+1}.$$

Put $H'_{1,0} = \cup_{j=-\infty}^{+\infty} (a'_{1,j}, b'_{1,j})$. For $n \geq 2$ put

$$H'_{n,0} = H'_{n-1,0} \cup \bigcup_{j=-\infty}^{+\infty} (a'_{n,j}, b'_{n,j}).$$

The set H_n' will consist of those lines which are parallel to $y = x$ and intersect the x-axis at a point belonging to $H_{n,0}'$, that is,

$$H_n' = \{(x,y) : \exists z \in H_{n,0}',\ y = x - z\}.$$

Put $H_{n,0}^+ = \cup_{j=-\infty}^{+\infty}(a_{n,j}, jd_n)$, $H_{n,0}^- = \cup_{j=-\infty}^{+\infty}(jd_n, b_{n,j})$,

$$H_1^+ = \{(x,y) : \exists z \in H_{1,0}^+, y = x - z\},$$

$$H_n^+ = \{(x,y) : \exists z \in H_{n,0}^+, y = x - z\} \setminus H_{n-1}'$$

and

$$H_1^- = \{(x,y) : \exists z \in H_{1,0}^-, y = x - z\},$$

$$H_n^- = \{(x,y) : \exists z \in H_{n,0}^-, y = x - z\} \setminus H_{n-1}'.$$

To define f_n first let $t_n = d_n/h_n$, and then put

$$f_n(x) = \begin{cases} t_n & \text{if } x \in H_n^+, \\ -t_n & \text{if } x \in H_n^-, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Since $\cup_{k=1}^{n-1}(H_k^+ \cup H_k^-) \subset H_{n-1}'$ and $H_{n-1}' \cap (H_n^+ \cup H_n^-) = \emptyset$ the sets $H_n^+ \cup H_n^-$ are disjoint for $n = 1, 2, \ldots$. Thus for any $x \in \mathbb{R}^2$ there exist at most one $n(x)$ such that $f_{n(x)}(x) \neq 0$. Hence $\sum_{n=1}^{\infty} f_n$ converges everywhere.

Remark 1 *Using that d_n/h_n, h_n/d_{n+1} are natural numbers, the definition of $a_{n,j}'$, $b_{n,j}'$, and of the sets H_n^+, H_n^-, H_n', it is easy to see that the stripes of the form $S_{n,j} = \{(x,y) : \exists z \in (a_{n,j}, b_{n,j}),\ y = x - z\}$ are either subsets of H_{n-1}' or are disjoint from H_{n-1}'.*

Remark 1 implies that if $u < v$ and $v - u$ is an integer multiple of d_n then the $+t_n$ and $-t_n$ values of f_n cancel each other on any segment of the form $\{(t,y) : u \le t \le v\} \subset \mathbb{R}^2 \setminus H_{n-1}'$, that is,

$$\int_u^v f_n(t,y)dt = 0 \text{ for any } y \in \mathbb{R}. \tag{3}$$

Using $|f_n| \le t_n$, (3), and the definition of the sets H_n^+, and H_n^- we infer that

$$\left| \int_u^v f_n(t,y)dt \right| \le h_n t_n = d_n \tag{4}$$

holds for any $u < v$ and $y \in \mathbb{R}$.

Assume that C is a figure. Put $F_n(C) = \int_C f_n d\lambda_2$ where λ_2 denotes the Lebesgue measure in $\mathbb{R}^2$. It is obvious that for any $y \in \mathbb{R}$ the intersection of C with the line $\{(t,y) : t \in \mathbb{R}\}$ consists of finitely many disjoint one-dimensional intervals. Denote the number of these intervals by $n(y)$. It is an easy exercise to verify that $\int_{-\infty}^{+\infty} n(y)dy \leq ||C||$. Using (4) we obtain

$$(5) \qquad |F_n(C)| = \left|\int_C f_n d\lambda_2\right| = \left|\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f_n(t,y)\chi_C(t,y)dt\right) dy\right| \leq$$

$$\left|\int_{-\infty}^{+\infty} d_n \cdot n(y)dy\right| \leq d_n \cdot ||C||,$$

where $\chi_C(t,y) = 1$ when $(t,y) \in C$ and $\chi_C(t,y) = 0$ otherwise.

Assume that $\eta > 0$ is given, Q is an η-regular interval and there exists $x_0 \in Q$ such that $x_0 \notin H_n'$. It is easy to see that $\mathrm{dist}(x_0, H_n^+ \cup H_n^-) > (c_n - 1)h_n/\sqrt{2}$. Therefore if $\mathrm{diam}(Q) < (c_n - 1)h_n/\sqrt{2}$ then $B(x_0, \mathrm{diam}(Q)) \cap (H_n^+ \cup H_n^-) = \emptyset$ and hence $\int_Q f_n d\lambda_2 = 0$. Thus from $\int_Q f_n d\lambda_2 \neq 0$, and $x_0 \in Q$ it follows that $(c_n - 1)h_n/\sqrt{2} \leq \mathrm{diam}(Q)$. Therefore the longer side of Q is of length at least $(c_n - 1)h_n/2$, and the η-regularity of Q implies that its shorter side is at least $\eta \cdot (c_n - 1)h_n/2$. Hence

$$(6) \qquad |Q| \geq \frac{(c_n - 1)^2 h_n^2 \eta}{4}, \qquad \text{and} \qquad |Q| \geq \eta \frac{(c_n - 1)h_n}{2}\frac{||Q||}{4}.$$

Assume that $S_{n,j}$ is a stripe defined in Remark 1. According to Remark 1, f_n is either identically zero on $S_{n,j}$, or $|f_n| = t_n$ on $S_{n,j} \cap (H_n^+ \cup H_n^-)$. In the first case $\int_{Q\cap S_{n,j}} f_n d\lambda_2 = 0$. In the second case, most of the t_n values of f_n are canceled by some $-t_n$ values. For ease of notation we illustrate this by working out some simple examples. If $S_{n,j}$ intersects the boundary of Q in two parallel segments then obviously the area of $Q \cap S_{n,j} \cap H_n^+$ equals the area of $Q \cap S_{n,j} \cap H_n^-$ and hence again $\int_{Q\cap S_{n,j}} f_n d\lambda_2 = 0$. Assume that $Q = [-(1/2),(1/2)] \times [0,1]$ and $S_{n,j} = S_{n,0} = \{(x,y) : \exists z \in (-h_n, h_n),\ y = x - z\}$. Then $S_{n,j} \cap Q$ can be split by the line $x = h_n$ into a parallelogram, denoted by P, and a triangle, denoted by E. It is again obvious that $\int_{Q\cap P} f_n d\lambda_2 = 0$. On the other hand the area of the triangle is small. Since its vertices are the points $(-h_n, 0)$, $(h_n, 0)$, and $(h_n, 2h_n)$ its area equals $2h_n^2$. Thus $|\int_{Q\cap S_{n,j}} f_n d\lambda_2| = |\int_{Q\cap E} f_n d\lambda_2| \leq 2h_n^2 t_n$. If $S_{n,j}$ contains the diagonal of Q then the estimations are somewhat more difficult than the above ones but one can see that $|\int_{Q\cap S_{n,j}} f_n d\lambda_2| \leq 8h_n^2 t_n$ holds (this estimation can be improved but for our purposes it is sufficient). In the general case arguments similar to the above ones can be used, the details are left to the reader. Finally one can obtain that

$$\left|\int_{Q\cap S_{n,j}} f_n d\lambda_2\right| \leq 8h_n^2 t_n. \tag{7}$$

It is easy to see that when n is fixed then the number of those $S_{n,j}$ for which $S_{n,j}\cap Q \neq \emptyset$ is no greater than $1+(||Q||/d_n)$. Since $F_n(Q) = \int_Q f_n d\lambda_2 = \sum_{j=-\infty}^{\infty} \int_{Q\cap S_{n,j}} f_n d\lambda_2$ by using (7) we obtain

$$|F_n(Q)| < 8t_n \cdot h_n^2 \left(1 + \frac{||Q||}{d_n}\right) = 8\frac{d_n}{h_n} \cdot h_n^2 \left(1 + \frac{||Q||}{d_n}\right) = 8h_n(||Q||+d_n). \tag{8}$$

Using (6) we infer

$$\frac{|F_n(Q)|}{|Q|} < \frac{8h_n(||Q||+d_n)}{|Q|} < 8\left(\frac{h_n||Q||}{|Q|} + h_n\frac{4d_n}{(c_n-1)^2h_n^2\cdot\eta}\right) \leq \tag{9}$$

$$8\left(\frac{h_n||Q||}{\eta\frac{(c_n-1)h_n}{2}\frac{||Q||}{4}} + \frac{4d_n}{(c_n-1)^2h_n\cdot\eta}\right) < 64\left(\frac{1}{\eta(c_n-1)} + \frac{d_n}{(c_n-1)^2h_n\cdot\eta}\right).$$

For an interval $Q \subset \mathbb{R}^2$ put $F(Q) = \sum_{n=1}^{\infty} F_n(Q)$. Observe that (1) and (8) imply that the series in the definition of F converges. Since a figure is the finite union of intervals $F(C)$ is well defined for any figure C.

Next we verify that F is the indefinite g-integral of f.

First we show that F is continuous. Assume that $\epsilon > 0$ is given. By using (1) choose $N_0 \in \mathbb{N}$ such that $\sum_{n=N_0}^{\infty} \frac{1}{\epsilon}d_n < \frac{\epsilon}{2}$. Since the functions f_n are bounded and measurable there exists an $\eta > 0$ such that if the figure C satisfies $|C| < \eta$ then $|\sum_{n=1}^{N_0-1} F_n(C)| < \frac{\epsilon}{2}$. Therefore if C is a given figure with $||C|| < \frac{1}{\epsilon}$ and $|C| < \eta$ then the above estimations and (5) imply

$$|F(C)| = \left|\sum_{n=1}^{\infty} F_n(C)\right| = \left|\sum_{n=1}^{N_0-1} F_n(C) + \sum_{n=N_0}^{\infty} F_n(C)\right| \leq$$

$$\frac{\epsilon}{2} + \sum_{n=N_0}^{\infty} ||C||d_n < \frac{\epsilon}{2} + \sum_{n=N_0}^{\infty} \frac{1}{\epsilon}d_n < \epsilon.$$

When $n > N_1$ we have $H'_{N_1} \cap (H_n^+ \cup H_n^-) = \emptyset$. Thus $f_n(x) = 0$ for any $n > N_1$, and $x \in H'_{N_1}$. Therefore $f(x) = \sum_{n=1}^{N_1} f_n(x)$, and $F(Q) = \sum_{n=1}^{N_1} F_n(Q)$ holds for any $x \in H'_{N_1}$ and interval $Q \subset H'_{N_1}$. The functions f_n are bounded, Lebesgue integrable functions and H'_{N_1} is open, therefore F is an almost everywhere differentiable function of an interval on H'_{N_1} and $F'(x) =$

$f(x)$ holds for almost every $x \in H'_{N_1}$. This also implies that $F'(x) = f(x)$ holds for almost every $x \in \cup_{n=1}^{\infty} H'_n$.

Assume that $x_0 \notin \cup_{n=1}^{\infty} H'_n$, and $\eta \in (0,1)$ are given, Q is an η-regular interval, and $x_0 \in Q$. Then (9) implies

$$\frac{|F_n(Q)|}{|Q|} < 64\left(\frac{1}{\eta(c_n-1)} + \frac{d_n}{(c_n-1)^2 h_n \cdot \eta}\right). \tag{10}$$

Assume that $\epsilon > 0$. By using (2) choose $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0}^{\infty} 64\left(\frac{1}{\eta(c_n-1)} + \frac{d_n}{(c_n-1)^2 h_n \cdot \eta}\right) < \epsilon. \tag{11}$$

It is easy to see that $x_0 \notin H'_{N_0-1}$ implies that there exists a neighborhood, U, of x_0 such that for $x \in U$ we have $f_n(x) = 0$ for $n = 1, \ldots, N_0 - 1$. Then $F(Q) = \sum_{n=N_0}^{\infty} F_n(Q)$ holds for any interval $Q \subset U$. Thus (10) and (11) imply

$$\frac{|F(Q)|}{|Q|} < \epsilon$$

for any η-regular interval $Q \subset U$ for which $x_0 \in Q$. Since $0 < \eta < 1$ and $\epsilon > 0$ was arbitrary we obtained $F'(x_0) = 0 = f(x_0)$.

Therefore we verified that F is a continuous function of an interval and $F'(x) = f(x)$ holds for every $x \in \mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ and for almost every $x \in \cup_{n=1}^{\infty} H'_n$.

Assume that $\epsilon > 0$, and the interval $A \subset \mathbb{R}^2$ are given. If F is differentiable at x and $F'(x) = f(x)$ choose $\delta(x) > 0$ such that

$$\left| f(x)|Q| - F(Q) \right| < \frac{\epsilon}{2|A|}|Q| \tag{12}$$

holds for any ϵ-regular interval satisfying $x \in Q$, and $Q \subset B(x, \delta(x))$.

If $x_0 \in \cup_{n=1}^{\infty} H'_n$ then there exists an $N_0 \in \mathbb{N}$ such that $x_0 \in H'_{N_0} \setminus H'_{N_0-1}$ (where put $H'_0 = \emptyset$). Choose $\delta_0(x_0) > 0$ such that $B(x_0, \delta_0(x_0)) \subset H'_{N_0}$. It is easy to see that from $x_0 \notin H'_{N_0-1}$ it follows that there exists $\delta_1(x_0) \in (0, \delta_0(x_0))$ for which $f_k(x) = 0$ holds for $k < N_0$ and $x \in B(x_0, \delta_1(x_0))$. It is also clear that $x \in H'_{N_0}$ implies $f_k(x) = 0$ for $k > N_0$. Thus $f(x) = f_{N_0}(x)$ when $x \in B(x_0, \delta_1(x_0))$.

The functions f_n are Lebesgue, and hence g-integrable. For each $n \in N$ using Proposition 2 with $\epsilon/2^{n+1}$ we can find a gage function δ'_n such that

$$\sum_{i=1}^{p} \left| f_n(x_i)|A_i| - F_n(A_i) \right| < \frac{\epsilon}{2^{n+1}} \tag{13}$$

holds for each δ'_n-fine subpartition $\{(A_i, x_i) : i = 1, \ldots, p\}$ of A.

If F is not differentiable at x_0 or $F'(x_0) \neq f(x_0)$ then $x_0 \in \cup_{n=1}^{\infty} H'_n$. Choose N_0 such that $x_0 \in H'_{N_0} \setminus H'_{N_0-1}$. Put $\delta(x_0) = \min\{\delta_1(x_0), \delta'_{N_0}(x_0)\}$.

Assume that $\{(A_i, x_i) : i = 1, \ldots, p\}$ is a δ-fine, ϵ-regular partition in A. Put

$$\Gamma = \{i \in \{1, \ldots, p\} : F \text{ is differentiable at } x_i \text{ and } F'(x_i) = f(x_i)\}.$$

Using (12) and (13) we obtain

$$\sum_{i=1}^{p} \left| f(x_i)|A_i| - F(A_i) \right| =$$

$$\sum_{i\in\Gamma} \left| f(x_i)|A_i| - F(A_i) \right| + \sum_{n=1}^{\infty} \sum_{\{i\notin\Gamma : x_i \in H'_n \setminus H'_{n-1}\}} \left| f_n(x_i)|A_i| - F_n(A_i) \right| \leq$$

$$\sum_{i\in\Gamma} \frac{\epsilon}{2|A|} \cdot |A_i| + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \leq \frac{\epsilon}{2|A|}|A| + \frac{\epsilon}{2} = \epsilon,$$

(when $\Psi = \emptyset$ then the "empty sum", $\sum_{\Psi}$, is defined to have value 0). This implies that f is g-integrable.

Finally we verify that $f \circ T$ is not g-integrable. Assume for a contradiction that $f \circ T$ is g-integrable. By Proposition 3 the indefinite g-integral of $f \circ T$, that is the interval function $(g)\int_A f \circ T$, is almost everywhere differentiable and its derivative at almost every x equals $f \circ T(x)$. We show that it is impossible.

For ease of notation instead of working with $f \circ T$, (ordinary) intervals, and (ordinary) figures we shall work with f, T-intervals, and T-figures respectively. If Q is a T-interval, then $T^{-1}(Q)$ is an interval. Put

$$G(Q) = (g)\int_{T^{-1}(Q)} f \circ T.$$

It is clear that if $f \circ T$ is g-integrable then G is an additive function of T-intervals which is almost everywhere differentiable with respect to T-intervals. Next we show that this leads to a contradiction.

Assume that $n \in \mathbb{N}$ is given and Q is a T-square such that two opposite sides of Q are on the lines $y = x - kd_n$, $y = x - (k+1)d_n$ and $Q \subset \mathbb{R}^2 \setminus H'_{n-1}$. Then the sides of Q are of length $d_n/\sqrt{2}$ and $|Q| = d_n^2/2$. Denote by S the closed stripe bounded by $y = x - (k+1)d_n - h_n$ and $y = x - (k+1)d_n$. Put $Q' = Q \cap S$. It is obvious that $|Q'| = d_n h_n/2$. From the definition of f_n and f it follows that $f_n(x) = f(x) = t_n$ on the interior of Q'. Thus

$$G(Q') = t_n|Q'| = \frac{d_n}{h_n} \cdot \frac{d_n h_n}{2} = \frac{d_n^2}{2} = |Q|.$$

Put $Q'' = Q \setminus Q'$. Then $G(Q) - G(Q'') = G(Q') = |Q|$. Therefore

$$\left|\frac{G(Q)}{|Q|} - \frac{G(Q'')}{|Q|}\right| = 1. \tag{14}$$

Next we state a proposition which is needed to complete the proof of the theorem. Then using the Proposition we complete the proof of the theorem. Although the proof of the Proposition is not too difficult, for the sake of completeness we finish the paper by proving the Proposition.

Proposition 4 *If x_0 is a point of density of $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ then one can find a sequence $\{Q_j : j = 1, 2, \ldots\}$ of T-squares such that $x_0 \in Q_j$, $diam(Q_j) \to 0$ as $j \to \infty$, and there exists $n(j)$ and $k(j)$ for which two opposite sides of Q_j are on the lines $y = x - k(j)d_{n(j)}$, $y = x - (k(j)+1)d_{n(j)}$ and $Q_j \subset \mathbb{R}^2 \setminus H'_{n(j)-1}$.*

Since almost every point of $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ is its point of density the conclusion of the Proposition holds for almost every x_0 in $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$.

Now we return to the proof of the theorem. One can define, as in the paragraph preceding the Proposition, the T-intervals Q'_j and Q''_j.

It is easy to see that the intervals Q''_j are

$$\frac{\frac{d_{n(j)}}{\sqrt{2}} - \frac{h_{n(j)}}{\sqrt{2}}}{\frac{d_{n(j)}}{\sqrt{2}}} = \frac{d_{n(j)} - h_{n(j)}}{d_{n(j)}} > \frac{1}{2}$$

regular. If G is differentiable at $x_0 \in \mathbb{R}^2$ then

$$\lim_{j\to\infty} \frac{G(Q_j)}{|Q_j|} = \lim_{j\to\infty} \frac{G(Q''_j)}{|Q''_j|} = G'(x_0). \tag{15}$$

Since

$$\lim_{j\to\infty} \frac{|Q''_j|}{|Q_j|} = \lim_{j\to\infty} \frac{d_{n(j)}(d_{n(j)} - h_{n(j)})}{d^2_{n(j)}} = 1$$

from (15) we obtain

$$\lim_{j\to\infty}\left|\frac{G(Q_j)}{|Q_j|} - \frac{G(Q''_j)}{|Q_j|}\right| =$$

$$\lim_{j\to\infty}\left|\frac{G(Q_j)}{|Q_j|} - \frac{G(Q''_j)}{|Q''_j|} \cdot \frac{|Q''_j|}{|Q_j|}\right| = 0$$

This contradicts (14). Thus at almost every points of $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ the T-interval function G is non-differentiable with respect to T-intervals.

Since it is easy to see that $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ is of positive Lebesgue measure we obtained a contradiction with the assumption that G is almost everywhere T-differentiable and this completes the proof of the Theorem assuming the Proposition.

Finally we verify the Proposition. Assume that x_0 is on the line $y = x - z_0$. From the construction of the sets H'_n it follows that if x_0 is a density point of $\mathbb{R}^2 \setminus \cup_{n=1}^{\infty} H'_n$ with respect to λ_2, then z_0 is a density point of $\mathbb{R} \setminus \cup_{n=1}^{\infty} H'_{n,0} = H''$ with respect to λ_1.

For every $n \in \mathbb{N}$ choose a j_n such that $z_0 \in [b'_{n,j_n-1}, a'_{n,j_n}] = I_n$. Since $x_0 \notin H'_n$ it is easy to see that $I_n \cap H'_{n,0} = \emptyset$. Put $I'_n = (b'_{n,j_n-1} + d_{n+1}, a'_{n,j_n} - d_{n+1})$. If $z_0 \in I_n \setminus I'_n$ then either z_0 is in $[b'_{n,j_n-1}, b'_{n,j_n-1} + d_{n+1}]$ or in $[a'_{n,j_n} - d_{n+1}, a'_{n,j_n}]$. Since the other case is similar we can assume that $z_0 \in [b'_{n,j_n-1}, b'_{n,j_n-1} + d_{n+1}]$. Observe that $(b'_{n,j_n-1} - d_{n+1}, b'_{n,j_n-1}) \subset (a'_{n,j_n-1}, b'_{n,j_n-1}) \subset H'_{n,0}$. Using $H'' \cap H'_{n,0} = \emptyset$ we have $\lambda_1([z_0 - 2d_{n+1}, z_0] \cap H'') < d_{n+1}$. Therefore if $z_0 \in I_n \setminus I'_n$ for infinitely many n's then z_0 cannot be a point of density of H''. Thus there exists an $N' \in \mathbb{N}$ such that for $n > N'$ we have $z_0 \in I'_n$.

Assume that for an n we have $z_0 \in I'_n$. Denote by k' the greatest integer for which $k' \cdot d_{n+1} \leq z_0$. It is easy to see that $z_0 \neq k' \cdot d_{n+1}$, and $z_0 \in (k' \cdot d_{n+1}, (k'+1) \cdot d_{n+1}) \subset I_n$. From $I_n \cap H'_{n,0} = \emptyset$ it follows that we can can find a T-square Q' such that $x_0 \in Q' \subset \mathbb{R}^2 \setminus H'_n$, and two opposite sides of Q' are on the lines $y = x - k'd_{n+1}$, $y = x - (k'+1)d_{n+1}$. The results of this and the preceding paragraph imply that one can find the sequence of T-squares required in the Proposition. This concludes the proof.

Remark 2 *The function f defined above is g-integrable but not g^*-integrable. Indeed, we verified that f is g-integrable. Since the rotation is a lipeomorphism, the assumption that f is g^*- integrable by the result of [1] would imply that $f \circ T$ is also g^*-integrable which is impossible since we showed that $f \circ T$ is not even g-integrable.*

References

[1] A. Novikov and W. F. Pfeffer, *An Invariant Riemann Type Integral Defined by Figures*, to appear in Proc. Amer. Math. Soc.

[2] W. F. Pfeffer, *Lectures on geometric integration and the divergence theorem*, Rend. Ist. Mat. Univ. Trieste, in press.

[3] W. F. Pfeffer, Letter sent to the author.