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PROPERTIES OF A GENERALIZED STIELTJES INTEGRAL DEFINED ON DENSE SUBSETS OF AN INTERVAL

1. Introduction

For real-valued functions f and g with domain including a closed interval $[a,b]$, we investigate an integral of f with respect to g defined on certain dense subsets of $[a,b]$ ($\Delta = \{M : a \text{ and } b \text{ belong to } M \text{ and } \overline{M} = [a,b]\}$). This concept was defined independently by Coppin [2] and Vance [4]. In 1972, we [1] showed the following: Suppose f and g are functions with domain $[a,b]$ and M belongs to Δ. Then f is g-integrable on M and $f|M$ and $g|M$ have no common points of discontinuity if and only if for any countable member M' of Δ which is a subset of M, f is g-integrable on M'.

2. Preliminary Definitions, Theorems and Notation

Herein, all functions are real-valued functions.

Throughout this paper, $[a,b]$ denotes a closed number interval and Δ denotes the set of all subsets of $[a,b]$ whose closure is $[a,b]$ and which contains a and b. In general, an interval (or an interval of M) is a set $[c,d]_M = [c,d] \cap M$ where M is a member of Δ, $[c,d]$ is a subinterval of $[a,b]$ and c and d belong to M.

Two intervals, A and B, are said to be nonoverlapping if and only if $A \cap B$ does not contain an interval. A nonempty collection of intervals is said to be nonoverlapping if and only if each two distinct members of the collection is nonoverlapping.

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If M is a member of Δ, then D is said to be a partition of M if and only if D is a finite collection of non-overlapping subintervals of M whose union is M. By $E(D)$ we mean the set of end points of members of D. D' is said to be a refinement of the partition D if and only if D' itself is a partition of M and $E(D) \subseteq E(D')$. We say that δ is a choice function on D if and only if δ is a function with domain D where $\delta(d) \in d$ for each d in D.

By the notation, $\Sigma(f, g, D, \delta)$, we mean

$$\Sigma(f,g,D,\delta) = \sum_{all\ [p,q]_M \in D} f(\delta([p,q]_M)) \cdot [g(q) - g(p)].$$

where D is a partition of a member of Δ, δ is a choice function on D, and f and g are functions with domain including $\cup D$.

Suppose that M is a member of Δ and f and g are functions with domain including M. Then f is said to be g-integrable on M if and only if there exists a number W (called "an integral of f with respect to g" and denoted by $\int_M fdg$) such that for each positive number ε, there is a partition D of M such that

$$|W - \Sigma(f,g,D',\delta)| < \varepsilon$$

for each refinement D' of D and each choice function δ on D'.

The following Stieltjes analogues will be used at various points in this paper. No proofs of these theorems are given because of similarity to their Stieltjes counterparts.

Theorem 2.1 *If f and g are functions with domain including $M \in \Delta$ and each of W_1 and W_2 is an integral of f with respect to g on M, then $W_1 = W_2$.*

Theorem 2.2 *If f and g are functions with domain including M a member of Δ and f is g-integrable on M, then $f|M$ and $g|M$ have no common discontinuities on either side of any member of M.*

Theorem 2.3 *Suppose f and g are functions with domain including M a member of Δ. Then the following two statements are equivalent:*

(a) f is g-integrable on M

(b) If $\epsilon > 0$, there is a partition D of M such that, if D' is a refinement of D, then

$$|\sum(f,g,D,\delta) - \sum(f,g,D',\delta')| < \epsilon$$

for each choice function δ on D and each choice function δ' on D'.

3. Results

The proof of the following theorem is made easier by using the Cauchy criterion of limits. From McLeod [3], we see that $\lim_{x\to z}(h|M)(x)$ exists if and only if for each $\epsilon > 0$ there exists some $\alpha > 0$ such that $|h(x_1) - h(x_2)| < \epsilon$ for all x_1 and x_2 in $(z-\alpha, z+\alpha) \cap M$. If we add the requirement that $x_1 < z < x_2$, an equivalent statement results that serves our purposes very well.

Theorem 3.1 *If f and g are functions with domain including $M \cup \{z\}$ where z belongs to $[a,b] - M$ and M is a member of Δ, and f is g-integrable on M, then $f|M$ or $g|M$ has a limit at z.*

Proof. Assume that f and g do not have limits at z.
Then, there is $k > 0$ such that for each $\alpha > 0$, we have

$$(1) \qquad |g(v) - g(u)| > k \text{ and } |f(x) - f(y)| > k$$

for some u, v, x, y in M where $z - \alpha < u < z < v < z + \alpha$, $u < x < z < y < v$.
Because f is g-integrable and (1), there is a partition D of M such that

(a) there is an element $[u,v]_M$ of D which has the property that $u < z < v$ and $|g(u) - g(v)| > k$ and there are elements x and y in M and between $u < x < z < y < v$ such that $|f(x) - f(y)| > k$,

(b) δ_1 and δ_2 are choice functions on D which are equal everywhere on D except that $\delta_1([u,v]_M) = x$ and $\delta_2([u,v]_M) = y$, and

(c) $|\int_M f dg - \sum(f, g, D, \delta)| < \frac{k^2}{2}$ where $\delta = \delta_1$ or $\delta = \delta_2$.

In (c), setting $\delta = \delta_1$ and $\delta = \delta_2$, and combining the two resultant inequalities, we obtain

$$(2) \qquad |f(x) - f(y)| \cdot |g(v) - g(u)| < k^2.$$

But, (2) is in contradiction with (a). Therefore, f or g has a limit at z.

Theorem 3.2 *Suppose that f and g are functions with domain including M such that*

(a) f is g-integrable on M,

(b) $M' \subseteq M$ where $M' \in \Delta$, and

(c) if z belongs to $M - M'$ and ϵ is a positive number, then there is an open interval s containing z such that $|f(x) - f(z)||g(v) - g(u)| < \epsilon$ where each of u, v, and x is in $s \cap M$, $u < z < v$, and $u \le x \le v$.

Then f is g-integrable on M', and $\int_M fdg = \int_{M'} fdg$.

Proof. Suppose $\epsilon > 0$. Since f is g-integrable on M, there is a partition D of M such that, if D' is a refinement of D, then

$$(3) \qquad |\int_M fdg - \sum(f,g,D',\delta)| < \frac{\epsilon}{2}$$

for each choice function δ on D'.

If $E(D) \subseteq M'$, the proof is straightforward. Thus, let us suppose that an element of D has an end point not belonging to M'.

Suppose $A = E(D) \cap (M')^c$ which can be written as

$$A = \{x_1, x_2, x_3, \ldots, x_N\}.$$

There is a collection $G = \{(r_i, s_i) : i = 1, 2, \ldots N\}$ of disjoint open intervals each of which contains exactly one element of A, contains no point of $E(D) \cap M'$, has end points in M', and, by hypothesis, if x_i belongs to A, then

$$(4) \qquad |f(x) - f(x_i)||g(v) - g(u)| < \frac{\epsilon}{2N}$$

for each u, v and x in $(r_i, s_i) \cap M$ where $u < x_i < v$, $u \le x \le v$ for $i = 1, 2, \ldots, N$.

Let D'' denote a refinement of D such that

$$E(D'') = E(D) \cup \{r_1, s_1, r_2, s_2, \ldots, r_N, s_N\}.$$

Let P denote a partition of M' such that $E(P) = E(D'') \cap M'$. Suppose that P' is a refinement of P. Let $[c_i, d_i]_{M'}$ denote the element of P' such that $c_i < x_i < d_i$ for $i = 1, 2, \ldots, N$.

From (4), since c_i,d_i and x_i are in $(r_i, s_i) \cap M$, we have

$$(5) \ |f(x)[g(d_i) - g(c_i)] - f(x_i)[g(x_i) - g(c_i)] - f(x_i)[g(d_i) - g(x_i)]| < \frac{\epsilon}{2N}$$

where x is any number in $[c_i, d_i]_{M'}$, $i = 1, 2, \ldots, N$. Since there are N elements in A, then, from (5) we have

$$(6) \quad |\sum_{i=1}^{N} f(x)[g(d_i) - g(c_i)] - \sum_{i=1}^{N} f(x_i)[g(x_i) - g(c_i)] - \sum_{i=1}^{N} f(x_i)[g(d_i) - g(x_i)]| < \frac{\epsilon}{2}.$$

Now, let δ'' be an arbitrary choice function on D'. In addition, let D''' denote a refinement of D such that $E(D''') = E(P') \cup E(D)$. Then, we have from (3)

$$|\int_M fdg - \sum(f, g, D''', \delta')| < \frac{\epsilon}{2} \tag{7}$$

for a choice function δ' on D''' where $\delta'([p,q]_M) = \delta''([p,q]_{M'})$ for each $[p,q]_M$ in P such that no point of A is in $[p,q]$ and $\delta'([c_i, x_i]_M) = \delta'([x_i, d_i]_M) = x_i, i = 1, 2, \ldots, N$.

Combining (7) and (6), we have

$$|\int_M fdg - \sum(f, g, P', \delta')| < \epsilon$$

for each choice function δ' on P'.

Therefore, by definition, f is g-integrable on M' and, by Theorem 2.1, $\int_M fdg = \int_{M'} fdg$.

Theorem 3.3 *If f and g are real-valued functions with domain including $[a,b]$ such that f is g-integrable on some uncountable member of Δ, then f is g-integrable on uncountably many members of Δ.*

Proof. Suppose that f is g-integrable on some uncountable member M of Δ and T is the collection of all members of Δ over which f is g-integrable. If f is g-integrable on $M - \{x\}$ for each $x \in M$, then T is uncountable.

Let $Q = \{z : z \in M \text{ and } f \text{ is not } g-\text{integrable on } M - \{z\}\}$.

If Q is void or countable, then T is uncountable. We show that Q cannot be uncountable.

Assume that Q is uncountable. If z is a member of Q, then the property described in part (c) of Theorem 3.2 where $M' = M - \{z\}$ cannot be true of f and g at z. This means there is a positive integer n such that, if s is an open interval containing z, then

$$|f(x) - f(z)||g(v) - g(u)| > \frac{1}{n}$$

for some u, v, and x in $s \cap M$ where $u < z < v$ and $u \leq x \leq v$. In fact, because Q is uncountable, we know that there exists some positive integer n where this inequality holds for infinitely many values of z. We denote this set by C.

Let N denote a positive integer so that $\frac{N}{n} > 1$.

There is a partition D of M such that, if D' is a refinement of D, then

$$(8) \qquad |\int_M fdg - \sum(f,g,D',\delta)| < \frac{1}{2}$$

for each choice function δ on D'. Since C is infinite, there is a member $[c,d]_M$ of D which contains infinitely many members of C. Let A denote a finite subset of $C \cap (c,d)$ such that A has exactly N elements. Let $A = \{z_1, z_2, ...z_N\}$ where $z_1 < z_2 < z_3 < ... < z_N$. And, let F denote a partition of $[c,d]_M$ such that $F = \{[c,u_1]_M, [u_1,v_1]_M, [v_1,u_2]_M, \ldots, [u_N,v_N]_M, [v_N,d]_M\}$. While constructing F, remembering the defining characteristics of C, we choose u_i,v_i, and x_i from M so that

$$|f(x_i) - f(z_i)||g(u_i) - g(v_i)| > \frac{1}{n}$$

where $u_i < z_i < v_i$ and $u_i < x_i < v_i$ for $i = 1, 2, \ldots, N$

Let P denote a refinement of D such that

$$E(P) = E(D) \cup \{u_1, v_1, u_2, v_2, \ldots, u_N, v_N\}.$$

Therefore, from (8), we have

$$(9) \qquad |\int_M fdg - \sum(f,g,P-F,\delta) - \sum_{i=1}^{N} f(r_i)[g(v_i) - g(u_i)]| < \frac{1}{2}$$

and

$$(10) \qquad |\int_M fdg - \sum(f,g,P-F,\delta) - \sum_{i=1}^{N} f(s_i)[g(v_i) - g(u_i)]| < \frac{1}{2}$$

where δ is some choice function on $P - F$, $r_i = x_i$ and $s_i = z_i$ if $[f(x_i) - f(z_i)][g(v_i) - g(u_i)] > 0$ or $r_i = z_i$ and $s_i = x_i$ if $[f(x_i) - f(z_i)][g(v_i) - g(u_i)] < 0$. Combining (9) and (10), we have

$$(11) \qquad 1 < \frac{N}{n} < \sum_{i=1}^{N} [g(r_i) - g(s_i)][f(v_i) - f(u_i)] < 1$$

Obviously, the preceding is a contradiction and so Q is countable. Therefore T is uncountable.

Corollary 3.4 *If f is g-integrable on a member M of Δ and no other member of Δ, then M is countable.*

Theorem 3.5 *If M is a countable member of Δ, then there are real-valued functions f and g with domain $[a,b]$ such that f is g-integrable on M and no other member of Δ.*

Proof. Suppose M is a countable set in Δ. Thus, we let $M - \{a,b\} = \{r_1, r_2, r_3, ...\}$ and $r_0 = b$.

Let ϕ and θ denote functions with domain M such that $\phi(a) = \theta(a) = 0$, $\phi(b) = \theta(b) = 1$, and

$$\theta(r_n) = \sum_{\text{all } p \text{ where } r_p < r_n} \frac{1}{2^p} \text{ for each } r_n \text{ in } M - \{a,b\} \tag{12}$$

and

$$\phi(r_n) = \sum_{\text{all } p \text{ where } r_p \leq r_n} \frac{1}{2^p} \text{ for each } r_n \text{ in } M - \{a,b\}. \tag{13}$$

Clearly, both ϕ and θ are increasing functions. As a consequence, $\phi(r_n^-)$, $\phi(r_n^+)$, $\theta(r_n^-)$ and $\theta(r_n^+)$ exist for each positive integer n. From the definition of ϕ, we know that $|\phi(r_n) - \phi(r_n^-)| = \frac{1}{2^n}$ and that ϕ is continuous on the right at each $r_n \in M - \{a,b\}$. Likewise, From the definition of θ, we know that $|\theta(r_n^+) - \theta(r_n)| = \frac{1}{2^n}$ and that θ is continuous on the left at each $r_n \in M - \{a,b\}$. We know that θ is continuous on the left at b.

Let f and g denote functions with domain the interval $[a,b]$ such that

$$f(x) = \begin{cases} \phi(x) & \text{if x belongs to M} \\ 2 & \text{if x belongs to [a,b] - M} \end{cases}$$

and

$$g(x) = \begin{cases} \theta(x) & \text{if x belongs to M} \\ 2 & \text{if x belongs to [a,b] - M} \end{cases}$$

Suppose $\epsilon > 0$.
There is a positive integer N such that $\frac{1}{2^N} < \frac{\epsilon}{4}$. Let

$$Q = \{r_i : i \text{ is a nonnegative integer and } i \leq N\}.$$

If r_i is an element of Q, since θ is continuous on the left at r_i, there is a number t_i of M such that no element of Q is between t_i and r_i, $t_i < r_i$, and

$$|\theta(r) - \theta(s)| < \frac{\epsilon}{6(N+2)} \text{ for all } r \text{ and } s \text{ in } [t_i, r_i,]_M.$$

Let D denote a partition of M where

$$E(D) = Q \cup \{a, b\} \cup \{t_i : i \text{ is a nonnegative integer and } i \leq N\}.$$

Suppose D' is a refinement of D. Let c and d denote any two consecutive elements of $Q \cup \{a\}$ with $c < d = r_i$, t_i is as described above, and s is the largest member of $E(D')$ which is less than d. The following four inequalities hold:

$$(14)\quad |\phi(x)[\theta(t_i) - \theta(c)] - \phi(x)[\theta(s) - \theta(c)]| = |\phi(x)||\theta(t_i) - \theta(s)|$$

$$(15)\quad < \frac{\epsilon}{6(N+2)}$$

where x is any element of $[c, t_i]_M$.

$$(16)\quad |\phi(u)[\theta(d) - \theta(s)]| < \frac{\epsilon}{6(N+2)}$$

where u is any element of $[s, d]_M$.

$$(17)\quad |\phi(v)[\theta(d) - \theta(t_i)]| < \frac{\epsilon}{6(N+2)}$$

where v is any element of $[t_i, d]_M$.

$$(18)\quad |\phi(x)[\theta(s) - \theta(c)] - \sum \phi(z)[\theta(w) - \theta(r)]| < [\phi(d^-) - \phi(c)]$$

where the sum is taken over all $[r, w]_M$ in D' where $c \leq r < w \leq s$, and z is any member of $[r, w]_M$.

Adding (14), (16), (17) and (18) we have

$$|\sum \phi(x)[\theta(q) - \theta(p)] - \sum \phi(y)[\theta(w) - \theta(r)]| < \frac{\epsilon}{2(N+2)} + [\phi(d^-) - \phi(c)]$$

where the first sum is taken over all $[p, q]_M$ in D such that $c \leq p < q \leq d$ and x is any element of $[p, q]_M$, and the second sum is taken over all $[r, w]_M$ in D' such that $c \leq r < w \leq d$ and y is any element of $[r, w]_M$. Therefore,

$$|\sum \phi(x)[\theta(q) - \theta(p)] - \sum \phi(y)[\theta(w) - \theta(r)]| < \frac{\epsilon}{2} + \sum [\phi(d^-) - \phi(c)]$$

where the first sum is taken over all $[p, q]_M$ in D where x is any member of $[p, q]_M$, the second sum is taken over all $[r, w]_M$ in D' where y is any member of $[r,w]_M$, and the third sum is taken over all consecutive c and d of Q. Note that

$$\sum[\phi(d^-)-\phi(c)]+\sum_{k=1}^{N}\frac{1}{2^k}=1$$

and, therefore,

$$\sum[\phi(d^-)-\phi(c)]=\frac{1}{2^N}<\frac{\epsilon}{4}$$

where the sum is over all c and d of Q. Now,

$$|\sum\phi(x)[\theta(q)-\theta(p)]-\sum\phi(y)[\theta(w)-\theta(r)]|<\epsilon$$

where the first sum is taken over all $[p,q]_M$ in D where x is any member of $[p,q]_M$ and the second sum is taken over all $[r,w]_M$ in D' where y is any member of $[r,w]_M$.

What has been shown is that, if $\epsilon>0$, there is a partition D of M such that, if D' is a refinement of D, then

$$|\sum(f,g,D,\delta)-\sum(f,g,D',\delta')|<\epsilon$$

for each choice function δ on D and each choice function δ' on D'. Remember that $f(x)=\phi(x)$ and $g(x)=\theta(x)$ if x belongs to M. By Theorem 2.3 this means that f is g-integrable on M.

Now, we must show that f is not integrable with respect to g on any other member of Δ.

Assume there is a member M' of Δ such that $M\neq M'$ and f is g-integrable on M'. There are two cases.

Case 1. M' contains a point not in M. Let $A=M'-M$. A is bounded below by a. Thus, let K denote the greatest lower bound of A. Since $f(a)=0$ and, by Theorem 2.2, both $f|M'$ and $g|M'$ cannot have right discontinuities at a, the number a cannot be a limit point of $M'-M$. Therefore, $K>a$. The members of M' less than K are members of M only. Thus, $f|M'(x)<1$ and $g|M'(x)<1$ for each x in $M'\cap[a,K)$

Assume K belongs to M'. Let $K\notin M$. Then $f(K)=g(K)$. We have that $f|M'$ and $g|M'$ are discontinuous on the left at K. By Theorem 2.2 , f is not g-integrable on M'. Now, let $K\in M$. Then $f|M'$ and $g|M'$ are discontinuous on the right at K. Again, by Theorem 2.2, f is not g-integrable on M'.

Now, assume K does not belong to M', by Theorem 3.1, f is not g-integrable on M' for neither f nor g has a limit at K.

Case 2. M' is a proper subset of M. There is a positive integer N such that r_N belongs to M but not M'. Recollect that on M, f and g are ϕ and θ, respectively. Earlier, we learned that ϕ and θ do not have limits at r_N. This contradicts Theorem 3.1. Thus, f is not g -integrable on M'.

Therefore f is not g-integrable on any member of Δ distinct from M.

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