Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 420–426

Krzysztof Ciesielski (email: kcies@wvnvms.bitnet), Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310

Sam B. Nadler, Jr., Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310

An absorption property for the composition of functions

1. Introduction

We investigate various classes of functions of a real or complex variable with respect to the following property: if $g \circ f$ is in a given class for some surjection f in the class, then g is in the class.

We begin by noting the following simple, fundamental result which is well known [10], but for which we include a proof since there does not seem to be a proof written down anywhere.

Theorem 1.1. *Let f be a continuous function from the reals, $\mathbb{R}^1$, onto $\mathbb{R}^1$. If g is a function from $\mathbb{R}^1$ into $\mathbb{R}^1$ such that $g \circ f$ is continuous, then g is continuous.*

Proof. Let $y_n, y \in \mathbb{R}^1$ such that $y_n \to y$. We will show that there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $g(y_{n_i}) \to g(y)$. This implies the continuity of g.

Choose points $p, q \in \mathbb{R}^1$ such that $p \leq y \leq q$ and $p \leq y_n \leq q$ for each n. Then, there are points $a \leq b$ such that $f(\{a,b\}) = \{p,q\}$. Then, since $f([a,b])$ is connected and $p, q \in f([a,b])$, $[p,q] \subset f([a,b])$. Thus, we have $y_n, y \in f([a,b])$ for each n. Hence, for each n, there exists $x_n \in [a,b]$ such that $f(x_n) = y_n$. Let $\{x_{n_i}\}_{i=1}^{\infty}$ be a convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ with $x_{n_i} \to x$. By the continuity of f, $f(x_{n_i}) \to f(x)$. Thus, since $f(x_{n_i}) = y_{n_i}$

Key Words: analytic functions, Baire functions, connectivity functions, continuous functions, Darboux functions, measurable functions, right absorption property

Mathematical Reviews subject classification: Primary 26A15, Secondary 54C99, 30A99

Received by the editors October 1, 1992

and $y_{n_i} \to y$, $f(x) = y$ and $f(x_{n_i}) \to y$. Hence, using continuity of $g \circ f$,

$$\lim_{i \to \infty} g(y_{n_i}) = \lim_{i \to \infty} (g \circ f)(x_{n_i}) = (g \circ f)(x) = g(y).$$

This finishes the proof. □

We remark that straightforward modifications of the proof just given show that Theorem 1.1 remains true when the domain of f is any metric semi-continuum (= any two points lie in a compact, connected subset) and the range of g is any topological space. Related results are in [7] and section IV of [8].

We also remark that f in Theorem 1.1 is required to map *onto* $\mathbb{R}^1$ since, otherwise, g could behave arbitrarily on $\mathbb{R}^1 \setminus f(\mathbb{R}^1)$.

The theorem above leads directly to the following general notion. Let $\mathcal{F}$ be a class of functions from a space X into itself such that at least one member of $\mathcal{F}$ is a surjection (= a map of X onto X.) We say that $\mathcal{F}$ has the *right absorption property* (abbreviated RAP) provided that if $g: X \to X$ is such that $g \circ f \in \mathcal{F}$ for some surjection $f \in \mathcal{F}$, then $g \in \mathcal{F}$.

Note that Theorem 1.1 says that the class of all continuous functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ has RAP. Also note that any class $\mathcal{F}$ of functions on a space X which is a group under composition has RAP (since $g = (g \circ f) \circ f^{-1}$ for any $g: X \to X$ and any $f \in \mathcal{F}$.)

We shall obtain some results about RAP. The classes of functions we consider are of general interest, and our results provide some interesting contrasts.

2. Results

We begin with the following classes of functions. Let $\mathcal{A}_1$ denote the class of all analytic functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ (i.e., $\mathcal{A}_1$ consists of all those C^∞ functions $f: \mathbb{R}^1 \to \mathbb{R}^1$ which are locally equal to their Taylor series.) Let $\mathcal{A}_2$ denote the class of all analytic functions from the complex plane, $\mathbb{C}$, into $\mathbb{C}$ [1, p. 24]. For each $n = 1, 2, \ldots$, let C_n denote the class of all continuous functions from Euclidean n-space, $\mathbb{R}^n$, into $\mathbb{R}^n$, and, for any k such that $1 \leq k \leq \infty$, let C_n^k denote the class of all functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ which are k times differentiable. (Note: $C_2^k \neq \mathcal{A}_2$ since, for C_2^k, we are using the Fréchet derivative, i.e., the matrix of partials.)

Theorem 2.1.

(a) $\mathcal{A}_2$ has RAP, but $\mathcal{A}_1$ does not have a RAP.

(b) C_n does not have RAP for $n > 1$ though C_1 has RAP.

(c) C_n^k does not have RAP for any $n = 1, 2, \ldots$ and for any $k = 1, 2, \ldots, \infty$.

PROOF OF (a). To prove that $\mathcal{A}_2$ has RAP, let $g: \mathbb{C} \to \mathbb{C}$ be such that $g \circ f \in \mathcal{A}_2$ for some $f \in \mathcal{A}_2$, where f maps $\mathbb{C}$ onto $\mathbb{C}$. Let $q \in \mathbb{C}$. We show that g is analytic at q. For this purpose, let $p \in \mathbb{C}$ be such that $f(p) = q$. We consider two cases.

Case 1: $f'(p) \neq 0$. Then there is an open neighborhood U of p such that f has an analytic inverse, h, on $W = f(U)$ [1, p.132]. Thus, since

$$g = (g \circ f) \circ h \quad \text{on } W \text{ and } \quad g \circ f \in \mathcal{A}_2,$$

g is analytic on W. Therefore, since W is an open neighborhood of q, we conclude that g is analytic at q.

Case 2: $f'(p) = 0$. First notice that f is not constant since f maps $\mathbb{C}$ onto $\mathbb{C}$. Hence, f is a continuous, open map [1, Corollary 1, p.132]. Thus, since $g \circ f$ is continuous on $\mathbb{C}$, g is continuous on $\mathbb{C}$ [11, Thms 9.2 and 9.4, p. 60]. Thus, to show that g is analytic at q, it suffices by [1, Thm 7, p. 124] to show that there is a region, Ω, such that $q \in \Omega$ and g is analytic on $\Omega \setminus \{q\}$. Since f is not constant, we see that the sets $f^{-1}(q)$ and $(f')^{-1}(0)$ have no accumulation point in $\mathbb{C}$ [1, p. 127]. Hence, there is a region, V, such that $p \in V$, $f'(z) \neq 0$ for any $z \in V \setminus \{p\}$, and $V \cap f^{-1}(q) = \{p\}$. Now, let $\Omega = f(V)$. Then, Ω is a region (since f is an open map), $q \in \Omega$, and, applying the argument in Case 1 to any $z \in V \setminus \{p\}$, we see that g is analytic on $\Omega \setminus \{q\}$. Therefore, g is analytic at q.

In view of what we have shown in Case 1 and Case 2, we have proved that $g \in \mathcal{A}_2$. Therefore, we have proved that $\mathcal{A}_2$ has RAP.

Now, to prove the second part of (a), simply let $f(x) = x^3$ and $g(x) = x^{1/3}$ for all $x \in \mathbb{R}^1$. Then, $f \in \mathcal{A}_1$, f maps $\mathbb{R}^1$ onto $\mathbb{R}^1$, $g \circ f \in \mathcal{A}_1$, and $g \notin \mathcal{A}_1$.

PROOF OF (b) AND (c). The second part of (b) is Theorem 1.1, and (c) for $n = 1$ is proved using f and g in the proof of the second part of (a). Finally, the first part of (b), as well as (c) for $n > 1$, is a consequence of Example 2.2 given below. □

Example 2.2. *For each $n > 1$, there exists C^∞ function f from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and a discontinuous function $g: \mathbb{R}^n \to \mathbb{R}^n$ such that $g \circ f$ is in C^∞.*

PROOF. We show this first for the case when $n = 2$.

Define $h: \mathbb{R}^1 \to \mathbb{R}^1$ by letting $h(x) = e^{-x^{-2}}$ for $x \neq 0$ and $h(0) = 0$. It is well known that h is C^∞ [4, p.40]. Now, define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by letting, for each $(x, y) \in \mathbb{R}^2$,

$$f(x,y) = \left(h(x) \cos \left[\frac{2\pi}{1+y^2} \right], h(x) \sin \left[\frac{2\pi}{1+y^2} \right] \right) \left[= h(x) e^{i2\pi/(1+y^2)} \right].$$

Clearly f is C^∞ map of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Also, note ($\star$) below which follows from simple calculations and which will be used to define the desired function:

$$(\star) \qquad \text{if } f(x_1, y_1) = f(x_2, y_2), \text{ then } (x_1^2, x_1^2 y_1^2) = (x_2^2, x_2^2 y_2^2).$$

Now, define g from $\mathbb{R}^2$ into $\mathbb{R}^2$ as follows. Let $(x, y) \in \mathbb{R}^2$. Then, since f maps $\mathbb{R}^2$ onto $\mathbb{R}^2$, there exists $(x_1, y_1) \in f^{-1}(x, y)$. Define

$$g(x, y) = (x_1^2, x_1^2 y_1^2).$$

Note that g is a function (i.e., depends only on (x, y)) since, by ($\star$), $g(x, y)$ is independent of the choice of $(x_1, y_1) \in f^{-1}(x, y)$. Since

$$(g \circ f)(z, w) = (z^2, z^2 w^2) \quad \text{for each} \quad (z, w) \in \mathbb{R}^2,$$

clearly $g \circ f$ is C^∞. However, g is not continuous since, as we now show, g is not continuous at $(e^{-1}, 0)$. For each $k = 1, 2, \ldots$, let $z_k = f(1, k)$. Then, since (by the formula for f)

$$z_k = e^{-1} e^{i2\pi/(1+k^2)} \quad \text{for each} \quad k,$$

we see that $z_k \to (e^{-1}, 0)$. However, since (by the formula for $g \circ f$)

$$g(z_k) = g(f(1, k)) = (1, k^2) \quad \text{for each} \quad k,$$

we see that the sequence $\{g(z_k)\}_{k=1}^{\infty}$ is unbounded. Therefore, g is not continuous at $(e^{-1}, 0)$. This completes the verifications of the example for the case when $n = 2$. To obtain such an example for any given $n > 2$, simply consider the functions

$$f_n = (f, id) \quad \text{and} \quad g_n = (g, id),$$

where $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ are as above and id is the identity map on $\mathbb{R}^{n-2}$. □

Remark. Let $\mathcal{F}$ be a class of functions from a space X into X such that the identity map of X is in $\mathcal{F}$; if there is a one-to-one, surjection $f \in \mathcal{F}$ such that $f^{-1} \notin \mathcal{F}$, then $\mathcal{F}$ does not have RAP. This general observation expresses the real reason that $\mathcal{A}_1$ does not have RAP (compare with the proof of the second part of (a) of Theorem 2.1). It can also be used to show that some other classes of functions do not have RAP. For example, it can be used to show that the class of density continuous functions and the class of $\mathcal{I}$-density continuous functions do not have RAP (in fact, each class contains a homeomorphism whose inverse is outside the class). See [9] and [3] for the former, and [2] for the latter.

For our next result, let $\mathcal{A}_2(\omega)$ denote the class of all continuous functions from $\mathbb{C}$ into $\mathbb{C}$ which are analytic at all but at most countably many points. Similarly, let $\mathcal{A}_1(\omega)$ denote the class of all continuous functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ which are analytic at all but countably many points. Moreover, for $f \in \mathcal{A}_1(\omega)$, we let $N(f)$ denote the set of all those points at which f is not analytic. (Note that $N(f)$ is closed in $\mathbb{R}^1$ since being analytic at a point is, by definition, a local property.)

Now, although $\mathcal{A}_2$ has RAP (Theorem 2.1(a)), the functions in Example 2.2 show that $\mathcal{A}_2(\omega)$ does not have RAP. The situation for A_1 and $\mathcal{A}_1(\omega)$ is the reverse – $\mathcal{A}_1$ does *not* have RAP (Theorem 2.1(a)), but $A_1(\omega)$ *does* have RAP as we now show.

Theorem 2.3. *The class $A_1(\omega)$ has RAP.*

PROOF. Let $g: \mathbb{R}^1 \to \mathbb{R}^1$ be such that $g \circ f \in \mathcal{A}_1(\omega)$ for some $f \in \mathcal{A}_1(\omega)$ where f maps $\mathbb{R}^1$ onto $\mathbb{R}^1$. Let

$$B = \{x \in \mathbb{R}^1 \setminus N(f): f'(x) = 0\} \quad \text{and} \quad A = B \cup N(f) \cup N(g \circ f).$$

Note that B is closed in $\mathbb{R}^1$. We will show that $f(A)$ is countable and that g is analytic on $\mathbb{R}^1 \setminus f(A)$. This will finish the proof since, by Theorem 1.1, g is continuous.

To see that $f(A)$ is at most countable, let V be a component of $\mathbb{R}^1 \setminus N(f)$. Then, V is open and either $B \cap V$ is at most countable or f is constant on $B \cap V$. In both cases, $f(B \cap V)$ is at most countable. Thus, since $\mathbb{R}^1 \setminus N(f)$ has at most countably many components and $B \subset \mathbb{R}^1 \setminus N(f)$, we conclude that $f(B)$ is at most countable. Thus, since the sets $N(f)$ and $N(g \circ f)$ are also at most countable, $f(A)$ is at most countable.

To show that g is analytic on $\mathbb{R}^1 \setminus f(A)$, let $q \in \mathbb{R}^1 \setminus f(A)$ and let $p \in \mathbb{R}^1$ such that $f(p) = q$. Then, $p \in \mathbb{R}^1 \setminus A$. In particular, f and $g \circ f$ are analytic at p and $f'(p) \neq 0$. Hence, there is an open neighborhood, U, of p, such that $U \subset \mathbb{R}^1 \setminus A$ and f has an analytic inverse, h, on $W = f(U)$. Thus, since

$$g = (g \circ f) \circ h \quad \text{on } W \text{ and } \quad U \cap N(g \circ f) = \emptyset,$$

g is analytic on W. Thus, g is analytic at q. □

Our next result is of a general nature. We give two specific applications of it in Corollary 2.5.

We adopt the following terminology. For a family, $\mathcal{F}$, of functions from a set X into X, we say that a set, C, is *hereditary for $\mathcal{F}$* provided that, for every $E \subset C$, the characteristic function of E, χ_E, belongs to $\mathcal{F}$.

Theorem 2.4. *Let $\mathcal{F}$ be a class of functions from a set X into X such that there exists $f \in \mathcal{F}$, f mapping X onto X, and a set S which is not hereditary for $\mathcal{F}$ but such that $f^{-1}(S)$ is hereditary for $\mathcal{F}$. Then, $\mathcal{F}$ does not have RAP.*

PROOF. Let f and S be as in the statement of the theorem, and let $E \subset S$ be such that $\chi_E \notin \mathcal{F}$. Put $g = \chi_E$. Then, $g \circ f \in \mathcal{F}$ since $g \circ f = \chi_{f^{-1}(E)}$ and $f^{-1}(E)$ is a subset of $f^{-1}(S)$ which is hereditary for $\mathcal{F}$. Thus, $\mathcal{F}$ does not have RAP. □

The following corollary is concerned with the class, $\mathcal{M}$, of all measurable functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ (the term measurable means Lebesgue measurable) and with the class, $\mathcal{B}$, of all functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ with the Baire property [5, p. 399], i.e., $f \in \mathcal{B}$ means that for every open set, U, in $\mathbb{R}^1$, $f^{-1}(U)$ has the Baire property. (A set $T \subset \mathbb{R}^1$ have the *Baire property* provided there exists an open set $W \subset \mathbb{R}^1$ such that the symmetric difference of T and W is of the first category in $\mathbb{R}^1$ [5, p. 87].)

Corollary 2.5. *Neither class $\mathcal{M}$ nor class $\mathcal{B}$ has RAP.*

PROOF. In what follows, C denotes the Cantor ternary set.

To prove the theorem for the class $\mathcal{M}$, let ψ denote a well-known homeomorphism of $[0,1]$ onto $[0,2]$ such that ψ maps C onto a set of measure one [4, Example 16, p. 98]. Extend ψ arbitrarily to a homeomorphism, f, from $\mathbb{R}^1$ onto $\mathbb{R}^1$. Then, $f \in \mathcal{M}$ and, letting $S = f(C)$, S is not hereditary for $\mathcal{M}$ while $C = f^{-1}(S)$ is hereditary for $\mathcal{M}$. Hence, by Theorem 2.4, $\mathcal{M}$ does not have RAP.

The proof for the class $\mathcal{B}$ is similar. Let T be a subset of C such that T is homeomorphic to the set of irrational numbers, $\mathbb{P}$. Then, T is a G_δ set. (In fact we can choose T as C without its end points.) Let $\phi: T \to \mathbb{P}$ be a homeomorphism onto $\mathbb{P}$. Extend ϕ to a Baire function f from $\mathbb{R}^1$ onto $\mathbb{R}^1$ such that $f(\mathbb{R}^1 \setminus T) = \mathbb{R}^1 \setminus \mathbb{P}$. Then: $f \in \mathcal{B}$, $\mathbb{P}$ is not hereditary for $\mathcal{B}$, and, since $f^{-1}(\mathbb{P}) = T \subset C$, $f^{-1}(\mathbb{P})$ is hereditary for $\mathcal{B}$. Hence, by Theorem 2.4, $\mathcal{B}$ does not have RAP. □

Two other classes of functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ which are of general interest are the class, $\mathcal{D}$, of all Darboux functions ($f \in D$ means the image, $f(I)$, of any interval I is connected) and the class, $\mathcal{K}$, of all connectivity functions ($f \in \mathcal{K}$ means the graph of the restriction of f to any interval is connected.) We note that $\mathcal{C}_1 \subset \mathcal{K} \subset \mathcal{D}$ and that $\mathcal{D} \neq \mathcal{K}$ [6, p. 131]. We have the following example.

Example 2.6. *Neither class $\mathcal{D}$ nor class $\mathcal{K}$ has RAP.*

PROOF. We will show this by giving an example of a function $f \in \mathcal{K}$, f mapping $\mathbb{R}^1$ onto $\mathbb{R}^1$, and a function $g: \mathbb{R}^1 \to \mathbb{R}^1$ such that $g \circ f \in \mathcal{K}$ and $g \notin \mathcal{D}$. The functions f and g are defined as follows:

$$f(x) = \begin{cases} x + |(1/x)\sin(1/x)| & x > 0 \\ 0 & x = 0 \\ x - |(1/x)\sin(1/x)| & x < 0 \end{cases} \qquad g(x) = \begin{cases} |x| & x \neq 0 \\ 1 & x = 0 \end{cases}$$

It is easy to see that f and g have the desired properties. □

References

[1] Lars V. Ahlfors, *Complex Analysis* (third edition), McGraw-Hill Book Company, New York, NY, 1979.

[2] V. Aversa and W. Wilczyński, Homeomorphisms preserving $\mathcal{I}$-density points, *Boll. Un. Mat. Ital., B(7)1* (1987), 275-285.

[3] Krzysztof Ciesielski and Lee Larson, The Space of Density Continuous Functions, *Acta Math. Acad. Sci. Hung.*, *58* (1991), 289-296.

[4] Bernard R. Gelbaum and John M.H. Olmsted, *Counterexamples in Analysis*, Holden-Day, Inc., San Francisco, 1964.

[5] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, NY, 1968.

[6] K. Kuratowski, *Topology*, Vol. II, Academic Press, New York, NY, 1968.

[7] Sam B. Nadler, Jr., Functional characterizations of trees, *Glasnik Mat.*, to appear.

[8] Sam B. Nadler, Jr., New problems in continuum theory, *Continuum Theory and Dynamical Systems* (Thelma West, editor), Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, NY, to appear.

[9] Jerzy Niewiarowski, Density-preserving homeomorphisms, *Fund. Math.*, *106* (1980),77-87.

[10] Hugh Thurston, Problem E3379, *Amer. Math. Monthly, 97* (1990), 342.

[11] Stephen Willard, *General Topology*, Addison-Wesley Publishing Company, Reading, Mass., 1970.