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ABSOLUTE INTEGRATION USING VITALI COVERS

1. Introduction

In the Henstock theory of integration, we use Riemann sums to define an integral. An immediate question is to prove the uniqueness of the integral. On the real line and for the ordinary covers, the uniqueness follows from the Heine-Borel covering theorem. For other cases, the situation is more complicated. At times, we have to use the category argument which is standard in the classical integration theory. It is interesting to note that when Kubota [1] proved the uniqueness of his integral he used the Vitali covering theorem and not the Heine-Borel covering theorem. The Vitali covers have also been used by McShane [3, p.89] to define a stochastic integral which he calls the belated integral. An advantage of using Vitali covers is that we do not need to prove the existence of partitions of a given interval. The $\mathcal{B}$-variational integral defined by Thomson [5, p.381] and the proximal integral defined by Sarkhel [4] are integrals of this sort, in which the existence of partitions is not necessary.

In this paper, we shall consider an absolute integral using Vitali covers, which includes McShane's nonstochastic Itô-belated integral, and prove that this absolute integral and the Lebesgue integral are equivalent. As a consequence, McShane's nonstochastic Itô-belated integral and the Lebesgue integral are equivalent, which gives an affirmative answer to the question (related to the above statement) posed by McShane in [3, p.91].

2. $\mathcal{B}^*$ integrals

The terminology used in this paper follows mainly Thomson's papers [5].

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Let R be the set of real numbers. Let $\mathcal{I}$ be the set of all closed intervals in R having a non-empty interior. An element $(I,x) \in \mathcal{I} \times R$ is called an interval-point pair. Let β be a collection of interval-point pairs. Then β is said to be a Vitali cover of a set $E \subset R$ if for each $\varepsilon > 0$ and any x in E there is an interval-point pair $(I,x) \in \beta$ such that $x \in I$ and the length $|I|$ of I is less than ε. Let $\mathcal{B}$ be a collection of Vitali covers of $[a,b]$. Then $\mathcal{B}$ is filtering if $\beta_1, \beta_2 \in \mathcal{B}$, then there is $\beta_3 \in \mathcal{B}$ with $\beta_3 \subset \beta_1 \cap \beta_2$. Let $D \subset \beta, \beta \in \mathcal{B}$. For brevity, we write $D = \{(I,x)\}$ where (I,x) denotes a typical interval-point pair in D. Then D is said to be a partial β-partition of $[a,b]$ if $\{I; (I,x) \in D\}$ is a finite collection of nonoverlapping subintervals of $[a,b]$. A partial β-partition $D = \{(I,x)\}$ of $[a,b]$ is a β-partition of $[a,b]$ if $\cup\{I; (I,x) \in D\} = [a,b]$.

Example (i). Let $\delta(x) > 0$ on $[a,b]$, and

$$\beta_\delta = \{([u,v],x); x \in [a,b], x \in [u,v] \subset (x - \delta(x), x + \delta(x))\}.$$

Then β_δ is a Vitali cover of $[a,b]$. Let $\mathcal{B}_H$ be the collection of all β_δ. Then $\mathcal{B}_H$ is filtering.

Example (ii). Let $\delta(x) > 0$ on $[a,b]$ and

$$\beta_\delta = \{([u,v],x); x \in [a,b], [u,v] \subset [x; x + \delta(x)), u = x\}.$$

Then β_δ is a Vitali cover of $[a,b]$. Let $\mathcal{B}_M$ be the collection of all β_δ. Then $\mathcal{B}_M$ is filtering.

Example (iii). Let δ be a positive constant, and

$$\beta_\delta = \{([u,v],x); x \in [a,b], x \in [u,v] \subset (x - \delta, x + \delta)\}.$$

Then β_δ is a Vitali cover of $[a,b]$. Let $\mathcal{B}_R$ be the collection of all β_δ. Then $\mathcal{B}_R$ is filtering.

For other interesting examples, see [5,pp.98-105].

Throughout this paper, a collection of Vitali covers of $[a,b]$ is always denoted by $\mathcal{B}$. We always assume that $\mathcal{B}$ is filtering. $(D)\sum$ denotes the sum over $D = \{(I,x)\}$ and $|I|$ denotes the length of an interval I.

Now we shall introduce the $\mathcal{B}^*$ integral and its properties.

Definition 1 *A function f defined on $[a,b]$ is said to be $\mathcal{B}^*$ integrable to A if for every $\varepsilon > 0$ there exist $\eta > 0$ and $\beta \in \mathcal{B}$ such that for any partial β-partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum |I| > b - a - \eta$, we have*

$$\left|(D)\sum f(x)|I| - A\right| < \varepsilon.$$

Note that the above definition is well-defined, in view of Vitali's covering theorem.

Denote $A = (\mathcal{B}^*)\int_a^b f$.

Most of the proofs of the following theorems are similar to those of the corresponding theorems for the Henstock integral, [2,pp.4-12].

Theorem 1 *The $\mathcal{B}^*$ integral is uniquely determined.*

PROOF. Let $\varepsilon > 0$. Suppose that there are A_1, A_2; η_1, η_2; β_1, β_2 satisfying Definition 1. Put $\eta = \min(\eta_1, \eta_2)$ and choose $\beta \in \mathcal{B}$ with $\beta \subset \beta_1 \cap \beta_2$. Then for any partial β-partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum |I| > b - a - \eta$ we have

$$\begin{aligned} |A_1 - A_2| &\le \left|A_1 - (D)\sum f(x)|I|\right| + \left|A_2 - (D)\sum f(x)|I|\right| \\ &< 2\varepsilon. \end{aligned}$$

We obtain $A_1 = A_2$.

Theorem 2 *If a function f defined on $[a,b]$ is Lebesgue integrable then f is $\mathcal{B}_H^*$ integrable on $[a,b]$, and*

$$(\mathcal{B}_H^*)\int_a^b f = (L)\int_a^b f,$$

where $\mathcal{B}_H$ is given in Example (i).

PROOF. Since a Lebesgue integrable function is Henstock integrable and the values of the two integrals agree [2, p.11], for every $\varepsilon > 0$ there is $\beta_{\delta_0} \in \mathcal{B}_H$ such that for any partial β_{δ_0}-fine partition $D = \{(I,x)\}$ of $[a,b]$ we have

$$(D)\sum \left|f(x)|I| - F(I)\right| < \varepsilon$$

where $F(I)$ denotes the Henstock integral of f over I.

Note that F is absolutely continuous on $[a,b]$. Then for $\varepsilon > 0$ there is $\eta > 0$ such that for any finite sequence $D' = \{I\}$ of nonoverlapping intervals with $(D')\sum |I| < \eta$, we have $(D')\sum |F(I)| < \varepsilon$. Consequently, for any partial β_{δ_0}-partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum |I| > b - a - \eta$, we have

$$\begin{aligned} \left|(D)\sum f(x)|I| - F(a,b)\right| &\le (D)\sum \left|f(x)|I| - F(I)\right| \\ &\quad + (D_1)\sum |F(I)| \\ &< 2\varepsilon \end{aligned}$$

where $(D_1)\sum$ denotes the sum over the complement intervals of $\cup\{I;(I,x)\in D\}$. Thus f is $\mathcal{B}_H^*$ integrable on $[a,b]$, and

$$(\mathcal{B}_H^*)\int_a^b f = (L)\int_a^b f.$$

Elementary properties of the $\mathcal{B}^*$ integral can be proved. However here we only prove that the Cauchy criterion holds and that if f is $\mathcal{B}^*$ integrable on $[a,b]$, then f is $\mathcal{B}^*$ integrable on each subinterval of $[a,b]$.

Lemma 3 (Cauchy Criterion) . *A function f is $\mathcal{B}^*$ integrable on $[a,b]$ if and only if for every $\varepsilon>0$ there are $\eta>0$ and $\beta\in\mathcal{B}$ such that for any two partial β-partitions $D=\{(I,x)\}$ and $D'=\{(I',x')\}$ with $(D)\sum|I|>b-a-\eta$ and $(D')\sum|I'|>b-a-\eta$, we have*

$$\left|(D')\sum f(x')|I'| - (D)\sum f(x)|I|\right| < \varepsilon.$$

PROOF. We shall only prove the sufficiency. Let ε_n be monotone decreasing to 0. Let η_n and β_n be given as in the above condition with given ε_n. We may assume that $\beta_{n+1}\subset\beta_n$ and $\eta_{n+1}\le\eta_n$, for each n. Let S_n denote $(D_n)\sum f(x)|I|$, where $D_n=\{(I,x)\}$ is a partial β_n-partition of $[a,b]$ with $(D_n)\sum|I|>b-a-\eta_n$. Here D_n, and thus S_n, are fixed. Note that if $m\ge n$, then D_m is a partial β_n-partition of $[a,b]$ and $(D_m)\sum|I|>b-a-\eta_m\ge b-a-\eta_n$. Hence

$$|S_n-S_m|<\varepsilon_n, \quad \text{if } m\ge n.$$

Therefore $A=\lim_{n\to\infty}S_n$ exists. Given $\varepsilon>0$, choose n with $\varepsilon_n\le\varepsilon$ and $|S_n-A|<\varepsilon$. Then for any partial β_n-partition $D=\{(I,x)\}$ of $[a,b]$ with $(D)\sum|I|>b-a-\eta_n$ we have

$$\begin{aligned}\left|(D)\sum f(x)|I|-A\right| &\le \left|(D)\sum f(x)|I|-S_n\right|+|S_n-A|\\ &< \varepsilon_n+\varepsilon\\ &< 2\varepsilon.\end{aligned}$$

Thus f is $\mathcal{B}^*$ integrable on $[a,b]$.

Theorem 4 *If f is $\mathcal{B}^*$ integrable on $[a,b]$, then f is $\mathcal{B}^*$ integrable on each closed subinterval of $[a,b]$.*

PROOF. Let $[c,d]\subset[a,b]$. Since f is $\mathcal{B}^*$ integrable on $[a,b]$, the Cauchy condition holds. Given $\varepsilon>0$, and let $\beta\in\mathcal{B}$ and $\eta>0$ be given as in the condition. We may assume that $|c-d|>\eta/2$ and $|a-b|-|c-$

$d| > \eta/2$. Take any two partial β-partitions $D_i = \{(I^{(i)}, x^{(i)})\}$ of $[c, d]$ with $(D_i)\sum |I^{(i)}| > |c-d| - \eta/2$, $i = 1, 2$. Take another partial β-partition $D_3 = \{(I^{(3)}, x^{(3)})\}$ of $[a, c] \cup [d, b]$ with $(D_3)\sum |I^{(3)}| > |c-a| + |b-d| - \eta/2$. Let $S_i = (D_i)\sum f(x^{(i)})|I^{(i)}|$, $i = 1, 2, 3$. Then $D_i \cup D_3$, for $i = 1, 2$, form partial β-partitions of $[a, b]$ with $(D_i \cup D_3)\sum |I| > b - a - \eta$. The Riemann sum of f over $D_i \cup D_3$ is $S_i + S_3$, for $i = 1, 2$. Therefore by the Cauchy condition we have

$$|S_1 - S_2| \leq |S_1 + S_3 - (S_2 + S_3)| < 2\varepsilon.$$

Thus f is $\mathcal{B}^*$ integrable on $[c, d]$.

Lemma 5 (Henstock's Lemma) . *Let f be $\mathcal{B}^*$ integrable on $[a, b]$. Then for every $\varepsilon > 0$, there is $\beta \in \mathcal{B}$ such that for any partial β-partition $D = \{(I, x)\}$ of $[a, b]$, we have*

$$(D)\sum \left| f(x)|I| - F(I) \right| < \varepsilon,$$

where $F(I)$ denotes the $\mathcal{B}^*$ integral of f over I.

PROOF. Let $\varepsilon > 0$. Then there exist $\eta > 0$ and $\beta \in \mathcal{B}$ such that for any partial β-partition $D_0 = \{(I, x)\}$ of $[a, b]$ with $(D_0)\sum |I| > b - a - \eta$, we have

$$\left| (D_0)\sum f(x)|I| - F(a, b) \right| < \varepsilon.$$

Let $D = \{(I, x)\}$ be any fixed partial β-partition of $[a, b]$. Let E_1 be the union of intervals I from D. Let E_2 be the closure of $[a, b]\backslash E_1$, say $E_2 = \{[a_i, b_i]\}_{i=1}^{N}$. Then f is $\mathcal{B}^*$ integrable on each $[a_i, b_i]$. For each i, there exist η_i with $0 < \eta_i \leq \eta/N$ and $\beta_i \in \mathcal{B}$ with $\beta_i \subset \beta$ such that for any partial β_i-partition $D_i = \{(I^{(i)}, x^{(i)})\}$ of $[a_i, b_i]$ with $(D_i)\sum |I^{(i)}| > b_i - a_i - \eta_i$, we have

$$\left| (D_i)\sum f(x^{(i)})|I^{(i)}| - F(a_i, b_i) \right| < \varepsilon/N.$$

For each i, choose a fixed D_i. Then

$$(D)\sum |I| + \sum_i (D_i)\sum |I^{(i)}| > |b-a| - \eta,$$

and $D' = \cup_i D_i \cup D$ is a partial β-partition of $[a, b]$. For simplicity, D' is denoted by $\{(I, x)\}$. Hence we have

$$\begin{aligned} \left| (D)\sum \{f(x)|I| - F(I)\} \right| &\leq \left| (D')\sum f(x)|I| - F(a, b) \right| \\ &\quad + \left| \sum_i (D_i)\sum f(x)|I| - F(a_i, b_i) \right| \\ &< \varepsilon + \varepsilon. \end{aligned}$$

Since D is any fixed partial β-partition of $[a,b]$ we have

$$(D)\sum\left|f(x)|I| - F(I)\right| < 4\varepsilon.$$

3. The Lebesgue integral

In this section, we shall prove that $\mathcal{B}^*$ integrals are more restrictive than the Lebesgue integral, when $\mathcal{B}$ has the ε-fine property.

Lemma 6 *Let f be $\mathcal{B}^*$ integrable on $[a,b]$. Let $\varepsilon > 0$. Then there are $\beta \in \mathcal{B}$ and $\eta > 0$ such that for any partial β-partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum|I| < \eta$ we have*

$$\left|(D)\sum f(x)|I|\right| < \varepsilon$$

and

$$\left|(D)\sum F(I)\right| < \varepsilon,$$

where $F(I)$ denotes the $\mathcal{B}^*$ integral of f over I.

PROOF. Let $\varepsilon > 0$. Then there are $\eta > 0$ and $\beta \in \mathcal{B}$ such that for any partial β-partition $D_0 = \{(I,x)\}$ of $[a,b]$ with $(D_0)\sum|I| > b-a-2\eta$, we have

$$\left|(D_0)\sum f(x)|I| - F(a,b)\right| < \varepsilon/2.$$

Let $D = \{(I,x)\}$ be a partial β-partition of $[a,b]$ with $(D)\sum|I| < \eta$. Let E be the union of intervals I from D and E_1 the closure of $[a,b]\backslash E$. Then the outer measure $|E_1|$ of E_1 is greater than $[b-a-\eta]-\eta$. Choose a partial β-partition $D' = \{(I',x')\}$ of $[a,b]$ with $x' \in E_1$ such that the union J of intervals I' from D' is in E_1 and $|E_1\backslash J| < \eta$. Hence

$$\begin{aligned} |[a,b]\backslash J| &= |(E\cup E_1)\backslash J| \\ &= |(E\cup(E_1\backslash J)| \\ &< \eta+\eta \\ |[a,b]\backslash(E\cup J)| &= |E_1\backslash J| \\ &< \eta. \end{aligned}$$

Therefore

$$\left|(D')\sum f(x')|I'| - F(a,b)\right| < \varepsilon/2$$

and

$$\left|(D)\sum f(x)|I| + (D')\sum f(x')|I'| - F(a,b)\right| < \varepsilon/2.$$

Hence

$$\left|(D)\sum f(x)|I|\right| < \varepsilon.$$

The second inequality follows from Henstock's Lemma and the above result.

Lemma 7 *Let f be $\mathcal{B}^*$ integrable on $[a,b]$ with primitive F. Then F is absolutely continuous on $[a,b]$.*

PROOF. By Lemma 6, for every $\varepsilon > 0$, there exist $\eta > 0$ and $\beta \in \mathcal{B}$ such that for any partial β-partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum |I| < \eta$, we have

$$\left|(D)\sum f(x)|I|\right| < \varepsilon.$$

Let $\{[a_i,b_i]\}$ be a finite sequence of nonoverlapping subintervals of $[a,b]$ with $\sum_i |b_i - a_i| < \eta$. Since f is $\mathcal{B}^*$ integrable on each $[a_i,b_i]$, there exist $\eta_i > 0$ and $\beta_i \in \mathcal{B}$ with $\beta_i \subset \beta$ such that for any partial β_i-partition $D_i = \{(I^{(i)},x^{(i)})\}$ of $[a_i,b_i]$ with $(D_i)\sum |I^{(i)}| > b_i - a_i - \eta_i$ we have

$$\left|(D_i)\sum f(x^{(i)})|I^{(i)}| - F(a_i,b_i)\right| < \varepsilon 2^{-i},$$

where $F(a_i,b_i)$ denotes the $\mathcal{B}^*$ integral of f over $[a_i,b_i]$. For each i fix a D_i and note that

$$\sum_i (D_i)\sum |I^{(i)}| \le \sum_i |b_i - a_i| < \eta,$$

and $D = \cup_i D_i$ is a partial β-partition of $[a,b]$. Hence

$$\left|\sum_i (D_i)\sum f(x^{(i)})|I^{(i)}|\right| < \varepsilon.$$

Therefore

$$\begin{aligned}\left|\sum_i F(a_i,b_i)\right| &\le \left|\sum_i (D_i)\sum f(x^{(i)})|I^{(i)}| - \sum_i F(a_i,b_i)\right| \\ &\quad + \left|\sum_i (D_i)\sum f(x^{(i)})|I^{(i)}|\right| \\ &< \varepsilon + \varepsilon.\end{aligned}$$

Thus F is absolutely continuous on $[a,b]$.

Let $\beta \in \mathcal{B}$ and $\delta(x) > 0$ on $[a,b]$. Then β is said to be δ-fine if $I \subset (x - \delta(x), x + \delta(x))$ whenever $(I,x) \in \beta$. A collection $\mathcal{B}$ is said to have the δ-fine property if for every $\delta(x) > 0$ on [a,b], there exists $\beta \in \mathcal{B}$, which is

δ-fine. If $\mathcal{B}$ has the δ-fine property and is filtering, then for every $\beta \in \mathcal{B}$ and for every $\delta(x) > 0$ on $[a, b]$, there exists $\beta_1 \in \mathcal{B}$ such that $\beta_1 \subset \beta$ and β_1 is δ-fine. Furthermore if G is open, then there exists $\beta \in \mathcal{B}$ such that $I \subset G$ whenever $x \in G$ and $(I, x) \in \beta$.

The collection $\mathcal{B}_R$ given in Example (iii) in Section 2 does not have the δ-fine property. Obviously, collections $\mathcal{B}_H$ and $\mathcal{B}_M$ have the δ-fine property.

A collection $\mathcal{B}$ of Vitali covers is said to have the ε-fine property if for any $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that $|I| < \varepsilon$ whenever $(I, x) \in \beta$.

It is clear that if $\mathcal{B}$ has the δ-fine property, then $\mathcal{B}$ has the ε-fine property. The converse is not true. The collection $\mathcal{B}_R$ has the ε-fine property.

Theorem 8 *Let $\mathcal{B}$ be filtering and have the ε-fine property. Let f be $\mathcal{B}^*$ integrable on $[a, b]$ with primitive F. Then $D_{\mathcal{B}}F(x) = f(x)$ for almost all $x \in [a, b]$, i.e., for almost all $x \in [a, b]$ and for every $\varepsilon > 0$, there exists $\beta_x \in \mathcal{B}$ such that*

$$\left| F(I) - f(x)|I| \right| < \varepsilon |I|$$

whenever $(I, x) \in \beta_x$, with $x \in I \subset [a, b]$.

PROOF. Since f is $\mathcal{B}^*$ integrable on $[a, b]$, by Henstock's Lemma, for every $\varepsilon > 0$ there is $\beta_0 \in \mathcal{B}$ such that for any partial β_0-partition $D = \{(I, x)\}$ of $[a, b]$, we have

$$(D) \sum \left| f(x)|I| - F(I) \right| < \varepsilon.$$

Let X be the set of points $x \neq a, b$ at which either $D_{\mathcal{B}}F(x)$ does not exist or, if it does, is not equal to $f(x)$. We shall prove that X is of measure zero.

From the definition of X we see that for every $x \in X$ there is a $\eta(x) > 0$ such that for every $\beta \in \mathcal{B}$, there exists one $(I, x) \in \beta, x \in I \subset [a, b]$ with

$$\left| F(I) - f(x)|I| \right| \geq \eta(x)|I|.$$

Fix n and let X_n denote the subset of X for which $\eta(x) \geq \frac{1}{n}$. Let A_n be the family of all interval-point pairs (I, x) with $x \in X_n \cap I$ such that $(I, x) \in \beta$ with $\beta \subset \beta_0$ and satisfying the above inequality. Then A_n is a Vitali cover of X_n. Here we use the fact that $\mathcal{B}$ has ε-fine property. Hence we can find interval-point pairs (I_k, x_k), $k = 1, 2, \ldots, m$ from A_n such that

$$|X_n| \leq \sum_{k=1}^{m} |I_k| + \varepsilon.$$

Therefore

$$\begin{aligned} |X_n| &\leq \sum_{k=1}^{m}\left\{\left|F(I_k)-f(x_k)|I_k|\right|/\eta(x_k)\right\}+\varepsilon \\ &< \varepsilon n+\varepsilon. \end{aligned}$$

Hence $|X_n| = 0$ and so $|X| = 0$.

Theorem 9 *Let $\mathcal{B}$ be filtering and have the ε-fine property. If f is $\mathcal{B}^*$ integrable on $[a,b]$, then f is Lebesgue integrable there and*

$$(\mathcal{B}^*)\int_a^b f = (L)\int_a^b f.$$

PROOF. By Lemma 7, the $\mathcal{B}^*$ primitive F of f is absolutely continuous on $[a,b]$. Hence the derivative $F'(x)$ exists for almost all $x \in [a,b]$. Therefore $F'(x) = f(x)$ for almost all $x \in [a,b]$, by Theorem 8. Hence f is Lebesgue integrable on $[a,b]$ and

$$(\mathcal{B}^*)\int_a^b f = (L)\int_a^b f.$$

By Theorems 9 and 2, we have

Corollary 10 *A function f defined on $[a,b]$ is Lebesgue integrable if and only if f is $\mathcal{B}_H^*$ integrable and the values of the two integrals agree.*

One of the referees pointed out that we did not investigate the dependence of the theory on $\mathcal{B}$. There is a great variety of integrals, and we only know that all $\mathcal{B}^*$ integrals, where $\mathcal{B}$ is filtering and has the ε-fine property, are more restrictive than Lebesgue integral, see Theorem 9. The referee posed the following question : Is a real restriction possible? We do not investigate this question in this paper. We thank the referee for his very constructive comments and interesting question.

4. McShane's nonstochastic Itô-belated integral

Now we shall introduce McShane's nonstochastic Itô-belated integral.

Definition 2 *A finite collection $D = \{(I,x)\}$ of interval-point pairs is called a partial partition of $[a,b]$ if $I, (I,x) \in D$, are nonoverlapping subintervals of $[a,b]$. Let $\delta(x) > 0$ almost everywhere on $[a,b]$. Then the partial partition $D = \{(I,x)\}$ of $[a,b]$ is called a partial belated δ-fine partition if for each $(I,x) \in D$, we have $I \subset [x, x+\delta(x))$. The point x need not belong to I.*

Definition 3 *(See [3, p.51].) A function f defined on $[a,b]$ is said to be Itô-belated integrable to A if for every $\varepsilon > 0$, there exist $\eta > 0$ and $\delta(x) > 0$ almost everywhere on $[a,b]$ such that for any partial belated δ-fine partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum|I| > b-a-\eta$ we have*

$$\left|(D)\sum f(x)|I| - A\right| < \varepsilon.$$

It is known that if f is Lebesgue integrable on $[a,b]$, then f is Itô-belated integrable there [3, pp.89-91]. McShane in [3, p.91] asks: Is the converse true? Next we shall show that the converse is true.

Lemma 11 *If f is Itô-belated integrable to A on $[a,b]$, then for every $\varepsilon > 0$, there exist $\eta > 0$ and $\delta(x) > 0$ on $[a,b]$ such that for any partial belated δ-fine partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum|I| > b-a-\eta$ we have*

$$\left|(D)\sum f(x)|I| - A\right| < \varepsilon.$$

PROOF. If f is Itô-belated integrable to A on $[a,b]$, then for every $\varepsilon > 0$, there exist $2\eta > 0$ and $\delta(x) > 0$ on $[a,b]\backslash B$ with B of measure zero such that for any partial belated δ-fine partition $D = \{(I,x)\}$ of $[a,b]$ with $(D)\sum|I| > b-a-2\eta$ we have

$$\left|(D)\sum f(x)|I| - A\right| < \varepsilon/2.$$

Let $B_n = \{x \in B; n-1 \le |f(x)| < n\}$, $n = 1,2,\ldots$. Let G_n be open with $B_n \subset G_n$ and $|G_n| < \min(\eta 2^{-n}, \varepsilon 2^{-n-1}n^{-1})$. If $x \in B_n$, define $\delta(x) > 0$ such that $(x-\delta(x), x+\delta(x)) \subset G_n$. Now if $D = \{(I,x); x \in B\}$ is a partial belated δ-fine partition of $[a,b]$, then

$$(D)\sum|I| < \sum_{n=1}^{\infty} \eta 2^{-n} = \eta.$$

Thus, if $D = \{(I,x)\}$ is a partial belated δ-fine partition of $[a,b]$ with

$$(D)\sum|I| > b-a-\eta$$

, then $(D_1)\sum|I| > b-a-2\eta$, where $D_1 = \{(I,x) \in D; x \notin B\}$. Hence D_1 is a partial belated McShane partition of $[a,b]$, and thus,

$$\left|(D_1)\sum f(x)|I| - A\right| < \varepsilon/2.$$

Therefore

$$\begin{aligned}\left|(D)\sum f(x)|I| - A\right| &< \varepsilon/2 + \left|(D\backslash D_1)\sum f(x)|I|\right| \\ &< \varepsilon/2 + \sum n\varepsilon 2^{-n-1}n^{-1} \\ &= \varepsilon.\end{aligned}$$

Theorem 12 *If f is Itô-belated integrable on $[a,b]$, then f is $\mathcal{B}_M^*$ integrable, and the values of two integrals agree.*

PROOF. Recall that $\mathcal{B}_M$ is given in Example (ii), Section 2. Let $\beta_\delta \in \mathcal{B}_M$. Note that if D is a partial β_δ-partition of $[a,b]$, then D is a partial belated δ-fine partition of $[a,b]$. Therefore, by Lemma 11, f is $\mathcal{B}_M^*$ integrable on $[a,b]$, and the values of two integrals agree.

Theorem 13 *If f is Itô-belated integrable on $[a,b]$, then f is Lebesgue integrable there, and the values of two integrals agree.*

PROOF. It follows from Theorems 9 and 12.

McShane established that the converse of Theorem 13 is true [3,pp.89-91], therefore we have

Corollary 14 *A function f is Itô-belated integrable on $[a,b]$ if and only if f is Lebesgue integrable there. Furthermore, the values of two integrals agree.*

By Theorems 12, 9 and Corollaries 14, 10, we have

Corollary 15 *The following four integrals are equivalent:*

The Itô-belated integral, the Lebesgue integral, the $\mathcal{B}_M^*$ integral and the $\mathcal{B}_H^*$ integral.

This means that, at least in the deterministic case, McShane's Itô-belated integral can be replaced by something altogether simple. This observation was pointed out by another referee. We thank the referee for this and other very constructive comments.

We remark that the Nonabsolute integration by using Vitali covers is being worked out. It will appear elsewhere.

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