Real Analysis Exchange
Vol. 18(2), 1992/93, pp. 400–408

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FUNCTIONS WITH POINTWISE DISCONTINUOUS RESTRICTIONS

Abstract

The paper contains comparisons of classes functions whose restrictions to some special sets of positive outer measure have continuity points, quasi-continuity points or are cliquish at some points.

Let $\mathbb{R}$ denote the set of reals and $\mathbb{N}$, the set of positive integers. A function $f : X \to \mathbb{R}$ ($\emptyset \neq X \subset \mathbb{R}$) is said to be quasicontinuous (cliquish) at a point $x \in X$ ([4], [6] (resp. [1])) if for every positive number r there is an open interval $I \subset (x - r, x + r)$ such that $I \cap X \neq \emptyset$ and $|f(t) - f(x)| < r$ for every $t \in I \cap X$ (resp. osc $f < r$ on $I \cap X$).

Let m (resp. m_e) denote Lebesgue measure (resp. outer Lebesgue measure) in $\mathbb{R}$. Denote by $C(f), C_q(f)$ and respectively $C_c(f)$ the set of all continuity points of a function $f : X \to \mathbb{R}$, the set of all quasicontinuity points of f and the set where f is cliquish. Let $\mathrm{cl}\,A$ denote the closure of a set A, int_X the interior in $X \neq \emptyset$ and for $Y \subset X$ ($Y \neq \emptyset$) and $f : X \to \mathbb{R}$ let $f|Y$ denote the restriction of f to Y. Put

$$\begin{aligned}
A_1 &= \{X \subset \mathbb{R} : m_e(X) > 0\},\\
A_2 &= \{X \in A_1 : X \text{ is an } F_\sigma\text{-set}\},\\
A_3 &= \{X \in A_1 : m_e(I \cap X) > 0 \text{ for every open interval } I \text{ with } I \cap X \neq \emptyset\},\\
A_4 &= A_3 \cap A_2,\\
A_5 &= \{X \in A_2 : X \text{ is closed}\};\\
A_6 &= A_5 \cap A_3,\\
A_7 &= \{X \subset \mathbb{R} : X \text{ is countable and } \mathrm{cl}\,X \in A_5\}, \text{ and}\\
A_8 &= \{X \subset \mathbb{R} : X \text{ is countable and } \mathrm{cl}\,X \in A_6\}.
\end{aligned}$$

*Supported by KBN research grant (1992–94) Nr 2 1144 91 01

Received by the editors July 30, 1992

In this paper I compare the following families of functions:

$$\begin{aligned}
H_j &= \{f : \mathbb{R} \to \mathbb{R};\ C(f|X) \neq \emptyset \text{ for every } X \in A_j\},\\
H_{jq} &= \{f : \mathbb{R} \to \mathbb{R};\ C_q(f|X) \neq \emptyset \text{ for every } X \in A_j\},\\
H_{jc} &= \{f : \mathbb{R} \to \mathbb{R};\ C_c(f|X) \neq \emptyset \text{ for every } X \in A_j\},\\
H_{0j} &= \{f : \mathbb{R} \to \mathbb{R};\ \operatorname{int}_X C(f|X) \neq \emptyset \text{ for every } X \in A_j\},\\
H_{0jq} &= \{f : \mathbb{R} \to \mathbb{R};\ \operatorname{int}_X C_q(f|X) \neq \emptyset \text{ for every } X \in A_j\}, \text{ and}\\
H_{0jc} &= \{f : \mathbb{R} \to \mathbb{R};\ \operatorname{int}_X C_c(f|X) \neq \emptyset \text{ for every } X \in A_j\},
\end{aligned}$$

$j = 1, 2, \ldots, 8$.
The family H_6 was introduced in [3] and the families H_1, H_2 were introduced and investigated in [2], where it is proven that $H_1 = H_2$.

The following remark is obvious.

Remark 1 *The following inclusions are true*

$H_{0j} \subset H_j$, $H_{0jq} \subset H_{jq}$, $H_{0jc} \subset H_{jc}$, $H_j \subset H_{jq} \subset H_{jc}$ *and* $H_{0j} \subset H_{0jq} \subset H_{0jc}$ *for* $j = 1, 2, \ldots, 8$. $H_1 \subset H_3 \subset H_4 \subset H_6$. $H_{1q} \subset H_{3q} \subset H_{4q} \subset H_{6q}$. $H_{1c} \subset H_{3c} \subset H_{4c} \subset H_{6c}$. $H_{01} \subset H_{03} \subset H_{04} \subset H_{06}$. $H_{01q} \subset H_{03q} \subset H_{04q} \subset H_{06q}$. $H_{01c} \subset H_{03c} \subset H_{04c} \subset H_{06c}$. $H_1 = H_2 \subset H_5 \subset H_6$. $H_{1q} \subset H_{2q} \subset H_{5q} \subset H_{6q}$. $H_{1c} \subset H_{2c} \subset H_{5c} \subset H_{6c}$. $H_2 \subset H_4 \subset H_5$. $H_{2q} \subset H_{4q} \subset H_{5q}$. $H_{2c} \subset H_{4c} \subset H_{5c}$.

Theorem 1 *We have* $H_{01c} = H_{02c} = H_{05c} = H_{07c} \subsetneqq H_{7c} \subsetneqq H_{1c} = H_{2c} = H_{5c} \subsetneqq H_{3c} = H_{4c} = H_{6c} = H_{8c} = H_{08c} = H_{06c} = H_{04c} = H_{03c}$.

PROOF. The inclusions $H_{01c} \subset H_{02c} \subset H_{05c}$ follow from Remark 1. We will show that $H_{05c} \subset H_{07c} \subset H_{01c}$. Let $f \in H_{05c}$ and let $X \in A_7$. Then $\operatorname{cl} X \in A_5$ and there is an open interval I such that $I \cap \operatorname{cl} X \neq \emptyset$ and $I \cap \operatorname{cl} X \subset C_c(f|\operatorname{cl} X)$. Consequently, $I \cap X \neq \emptyset$ and $I \cap X \subset C_c(f|X)$. This proves the inclusion $H_{05c} \subset H_{07c}$. Now let $f \in H_{07c}$ and let $X \in A_1$. There is a countable set $Y \subset X$ such that

$$\operatorname{cl}(\{(t, f(t));\ t \in Y\}) \supset \{(t, f(t));\ t \in X\}. \tag{1}$$

Since $Y \in A_7$ and $f \in H_{07c}$, there is an open interval I such that $I \cap Y \neq \emptyset$ and $I \cap Y \subset C_c(f|Y)$. Fix $x \in I \cap Y$, $r > 0$, and an open interval J containing x. There is an open interval $K \subset I \cap J$ such that $K \cap Y \neq \emptyset$ and $\operatorname{osc} f < r/2$ on $K \cap Y$. By (1), $\operatorname{osc} f \leq r/2 < r$ on $K \cap X$. Thus $I \cap Y \subset C_c(f|X)$. Since $C_c(f|X)$ is closed in X, we have $I \cap X \subset C_c(f|X)$. This proves the inclusion $H_{07c} \subset H_{01c}$. So, $H_{01c} = H_{02c} = H_{05c} = H_{07c}$.

Let $F \subset [0,1]$ be a nowhere dense set belonging to A_6. In each component I_n of the set $\mathbb{R} \setminus F$ we find a Cantor set F_n of measure zero. Let $F_n = F_{n,1} \cup F_{n,2}$, where all sets $F_{n,1}$, $F_{n,2}$ are disjoint non Borel, dense in F_n. Let

$$f(x) = \begin{cases} 1/n & \text{for} \quad x \in F_{n,1},\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases},$$

and

$$g(x) = \begin{cases} 1 & \text{for} \quad x \in F_{n,1},\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Then $f \in H_{1c} \setminus H_{05c}$, $f \in H_{5c} \setminus H_{7c}$, and $g \iota H_{6c} \setminus H_{5c}$. The proof of the inclusion $H_{7c} \subset H_{5c}$ is similar as the proof of the inclusion $H_{07c} \subset H_{01c}$. So, by Remark 1, $H_{07c} \subsetneqq H_{7c} \subsetneqq H_{5c}$.

Now we shall prove that $H_{1c} = H_{2c} = H_{5c}$. By Remark 1, $H_{1c} \subset H_{2c} \subset H_{5c}$. Let $f \in H_{5c}$ and let $X \in A_1$. Then $\operatorname{cl} X \in A_5$ and $C_c(f|\operatorname{cl} X) \neq \emptyset$. If the set $C_c(f|\operatorname{cl} X)$ is dense in the set $I \cap \operatorname{cl} X$ for an open interval I with $I \cap X \neq \emptyset$, then $I \cap \operatorname{cl} X \subset C_c(f|\operatorname{cl} X)$ since the set $C_c(f|\operatorname{cl} X)$ is closed. Consequently, in this case $I \cap X \subset C_c(f|X)$. Assume that the set $C_c(f|\operatorname{cl} X)$ is nowhere dense in $\operatorname{cl} X$. Then

$$\operatorname{cl} X \setminus C_c(f|\operatorname{cl} X) = \bigcup_{n \in \mathbb{N}} (I_n \cap \operatorname{cl} X),$$

where I_n are mutually disjoint open intervals such that $I_n \cap \operatorname{cl} X \neq \emptyset$ for $n \in \mathbb{N}$. Since $f|\operatorname{cl} X$ is not cliquish at any point $u \in \operatorname{cl} X \setminus C_c(f|\operatorname{cl} X)$, we have $m(I_n \cap \operatorname{cl} X) = 0$ for $n = 1, 2, \ldots$. Thus there is a point $x \in X \cap C_c(f|\operatorname{cl} X)$ and the restricted function $f|X$ is cliquish at x. So $C_c(f|X) \neq \emptyset$ and the equalities $H_{1c} = H_{2c} = H_{5c}$ are proved.

For the proofs of the remaining equalities we remark that the inclusions $H_{3c} \subset H_{4c} \subset H_{6c}$, $H_{03c} \subset H_{04c} \subset H_{06c}$ and $H_{0jc} \subset H_{jc}$ for $j = 1, \ldots, 8$ follow from Remark 1. Let $f \in H_{6c}$ and let $X \in A_3$. Then $\operatorname{cl} X \in A_6$ and $C_c(f|\operatorname{cl} X) = \operatorname{cl} X$. Consequently, $C_c(f|X) = X$ and $f \in H_{03c}$. So the inclusion $H_{6c} \subset H_{03c}$ is valid, and $H_{3c} = H_{4c} = H_{6c} = H_{03c} = H_{04c} = H_{06c}$.

Analogously, if $f \in H_{6c}$ and $X \in A_8$, then $\operatorname{cl} X \in A_6$ and $C_c(f|\operatorname{cl} X) = \operatorname{cl} X$. Thus $C_c(f|X) = X$ and $f \in H_{08c}$. So $H_{6c} \subset H_{08c} \subset H_{8c}$. Suppose that there is a function $f \in H_{8c} \setminus H_{6c}$. Then there is a set $X \in A_6$ such that $C_c(f|X) = \emptyset$. There is a countable set $Y \subset X$ such that (1) is satisfied. Since $f \in H_{8c}$, the set $C_c(f|Y)$ is nonempty. If $x \in C_c(f|Y)$, then $x \in X$ and by(1), $x \in C_c(f|Y)$. This is contrary with the equality $C_c(f|X) = \emptyset$. So $H_{8c} \subset H_{6c}$ and the proof is completed.

Theorem 2 *We have*
$H_{01} = H_{02} = H_{05} = H_{01q} = H_{02q} = H_{05q} = H_{07} = H_{07q} = H_7 = H_{7q} \subsetneqq H_{03} = H_{04} = H_{06} = H_{08} = H_8 = H_{03q} = H_{04q} = H_{06q} = H_{08q} = H_{8q}$.

PROOF. The inclusions $H_{01} \subset H_{02} \subset H_{05}$, $H_{01q} \subset H_{02q} \subset H_{05q}$, $H_{03} \subset H_{04} \subset H_{06}$, $H_{03q} \subset H_{04q} \subset H_{06q}$, $H_{0j} \subset H_{0jq}$ for $j = 1, \dots, 8$. $H_{07} \subset H_7$, $H_{08} \subset H_8$, $H_{07q} \subset H_{7q}$ and $H_{08q} \subset H_{8q}$ follow from Remark 1.

Let $f \in H_{05}$ and let $X \in A_1$. Then $\operatorname{cl} X \in A_5$ and there is an open interval I such that $I \cap \operatorname{cl} X \neq \emptyset$ and $I \cap \operatorname{cl} X \subset C(f|\operatorname{cl} X)$. Consequently, $I \cap X \neq \emptyset$ and $I \cap X \subset C(f|X)$. This proves that $f \in H_{01}$. So $H_{05} \subset H_{01}$ and $H_{01} = H_{02} = H_{05}$. The proof of the inclusion $H_{05q} \subset H_{01q}$ is the same and hence $H_{01q} = H_{02q} = H_{05q}$. Similarly we can prove that $H_{05} \subset H_{07}$ and $H_{05q} \subset H_{07q}$. So $H_{05} \subset H_7$ and $H_{05q} \subset H_{7q}$. Suppose that $f \in H_7$ and $X \in A_5$. For an indirect proof assume that $\operatorname{int}_X C(f|X) = \emptyset$. There is a countable set $Y \subset X \setminus C(f|X)$ such that condition (1) from the proof of Theorem 1 is satisfied. Then $Y \in A_7$ and $\emptyset \neq C(f|Y) \subset C(f|X)$ contrary to $C(f|Y) \subset Y \subset X \setminus C(f|X)$. So $H_7 \subset H_{05}$. The proof of the inclusion $H_{7q} \subset H_{05q}$ is similar. So $H_{05} = H_7$ and $H_{05q} = H_{7q}$. Now we shall show that $H_{06} = H_{06q}$. It suffices to prove that $H_{06q} \subset H_{06}$. In the proof of this inclusion an idea from Natkaniec is used. (See [5].) Let $f \in H_{06q}$ and let $X \in A_6$. Suppose that $\operatorname{int}_X C(f|X) = \emptyset$. Let $I_{n,k} = ((k-1)/2^n, k/2^n)$ for $n = 1, 2, \dots$ and $k = 0, \pm 1, \pm 2, \dots$. There are indices n_1, k_1 such that $m(I_{n_1,k_1} \cap X) > 0$ and $I_{n_1,k_1} \cap X \subset C_q(f|X)$. Let $x_{1,1} \in X \cap I_{n_1,k_1}$ be a point at which $f|X$ is not continuous. Then $a_1 = \operatorname{osc}(f|X)(x_{1,1}) > 0$. There is an open interval J_1 such that $x_{1,1} \in J_1 \subset \operatorname{cl} J_1 \subset I_{n_1,k_1}$ and $m(J_1) < m(X \cap I_{n_1,k_1})/8$. Let $U_1 = J_1 \cap \operatorname{int}_X(\{x \in X;\ |f(x) - f(x_{1,1})| < a_1/2\})$. Then the set U_1 is open in X and $x_{1,1} \notin U_1$.

In the second step we consider open intervals I_{n_1+1,k_2}, I_{n_1+1,k_2+1} such that $\operatorname{cl} I_{n_1,k_1} = \operatorname{cl} I_{n_1+1,k_2} \cup \operatorname{cl} I_{n_1+1,k_2+1}$. If $m((X \setminus \operatorname{cl} U_1) \cap I_{n_1+1,k_2+j}) > 0$, $j = 0$ or 1, there is a point $x_{2,j+1} \in (I_{n_1+1,k_2+j} \cap (X \setminus \operatorname{cl} U_1)) \setminus C(f|X))$. For $j = 0, 1$ set $a_{2,j} \operatorname{osc}(f|X)(x_{2,j+1})$ and let

$$U_{2,j} = J_{2,j} \cap \operatorname{int}_X(\{x \in X;\ |f(x) - f(x_{2,j+1})| < a_{2,j}/2\}),$$

where $J_{2,j}$ is an open interval such that $x_{1,1} \notin \operatorname{cl} J_{2,j}$, $x_{2,j} \in J_{2,j} \subset \operatorname{cl} J_{2,j} \subset I_{n_1+1,k_2+j}$, and $m(J_{2,j}) < m(X \cap I_{n_1+1,k_2+j})/(2 \cdot 8^2)$. In the n^{th} step we consider open intervals $I_{n_1+n,k_n}, I_{n_1+n,k_n+1}, \dots, I_{n_1+n,k_n+2^n-1}$ such that $\operatorname{cl} I_{n_1,k_1} = \operatorname{cl}(I_{n_1+n,k_n}) \cup \dots \cup \operatorname{cl}(I_{n_1+n,k_n+2^n-1})$, and we find points

$$x_{n,j+1} \in (I_{n_1+n,k_n+j} \cap (X \setminus \bigcup_{i=1}^{n-1} \bigcup_{k=0}^{2^i-1} \operatorname{cl} U_{i,k})) \setminus C(f|X)$$

(whenever the last set on the right side is nonempty). Next we find open intervals $J_{n,j}$, $j = 0, 1, \dots, 2^{n-1} - 1$, such that

$$x_{n,j+1} \in J_{n,j} \subset \operatorname{cl} J_{n,j} \subset I_{n_1+n,k_n+j},$$

$$x_{i,1} \notin \mathrm{cl}\, J_{n,j} \text{ for } i < n \text{ and } 1 < 2^i,$$

and

(2) $$m(J_{n,j}) < m(X \cap I_{n_1+n,k_n+j})/(2^{n-1} \cdot 8^n),$$

and set

$$U_{n,j} = J_{n,j} \cap \mathrm{int}_X(\{x \in X;\ |f(x) - f(x_{n,j+1})| < a_{n,j}/2\}),$$

where $a_{n,j} = \mathrm{osc}(f|X)(x_{n,j+1}) > 0$. The possibility of finding such points $x_{n,k}$ follows from the inclusion $X \cap I_{n_1,k_1} \subset C_q(f|X)$ and from the fact that $\mathrm{osc}(f|X)(x_{n-1,j+1}) = a_{n-1,j} > 0$. Observe that $x_{n,j+1} \notin U_{n,j}$ for $j = 0, 1, \ldots, 2^n - 1$ and that for each

$$x \in X \setminus \bigcup_{i=1}^{n} \bigcup_{k=1}^{2^i-1} \mathrm{cl}\, U_{i,k}$$

there is an index $j \leq 2^n - 1$ such that $|x - x_{n,j+1}| \leq 2^{-n}$. Let

$$Y = \mathrm{cl}\left(X \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n-1} U_{n,k}\right).$$

Fix $x = x_{n,j+1}$ where $j < 2^n$, and an open interval I containing x. There are an open interval $I_{n_1+m,k_m+j'}$ and a point $x_{m,j'+1}$ $(m > n)$ such that

$$x_{m,j'+1} \in I_{n_1+m,k_m+j'} \subset I.$$

It follows from our construction and from (2) that

$$\begin{aligned} m(I \cap Y) &\geq m(I_{n_1+m,k_m+j'} \cap Y) \\ &\geq m(I_{n_1+m,k_m+j'} \cap X) - \sum_{k=1}^{\infty} m(I_{n_1+m,k_m+j'} \cap X)/2^{k-1} \cdot 8^k \\ &= m(I_{n_1+m,k_m+j'} \cap X) - 2m(I_{n_1+m,k_m+j'} \cap X)/15 \\ &= 13m(I_{n_1+m,k_m+j'} \cap X)/15 > 0. \end{aligned}$$

Let $Z = \{x \in Y;\ m(I \cap Y) > 0 \text{ for each open interval } I \ni x\}$. Then $Z \in A_6$ and $x_{n,j+1} \in Z \setminus C_q(f|Z)$ for $n \in \mathbb{N}$ and $j < 2^n$ contrary to $f \in H_{06q}$. So $H_{06} = H_{06q}$.

Now we show that $H_{05q} \subset H_{05}$. Let $f \in H_{05q}$ and let $X \in A_5$. If $X \in A_6$, then by the equality $H_{06} = H_{06q}$, there is an open interval I such that $I \cap X \neq \emptyset$ and $I \cap X \subset C(f|X)$. Suppose that $X \in A_5 \setminus A_6$ and that $\mathrm{int}_X(C(f|X)) = \emptyset$. Let I be an open interval such that $I \cap X \neq \emptyset$ and $I \cap X \subset$

$C_q(f|X)$. Let $Y = \{x \in X;\ m(J \cap X) > 0$ for every open interval $J \ni x\}$. Then $Y \in A_6$ and $m(X \setminus Y) = 0$. Since $X \in A_5 \setminus A_6$ and $\mathrm{int}_X(C(f|X)) = \emptyset$, we can assume that the endpoints of the interval I belong to $\mathbb{R} \setminus X$ and that $\mathrm{cl}\,(X \cap (I \setminus Y)) = (X \cap (I \setminus Y)) \cup (I \cap Y)$. There is a sequence of disjoint nonempty open sets U_n such that $U_n \cap (X \setminus Y)$ is a nonempty closed set for $n \in \mathbb{N}$ and $I \cap (X \setminus Y) = \bigcup_n (U_n \cap (X \setminus Y))$. For every n the set $V_n = U_n \cap (X \setminus Y)$ is closed, $V_n \subset C_q(f|V_n)$ and $\mathrm{int}_{V_n}(C(f|V_n)) = \emptyset$. So by Natkaniec's theorem ([5]) for $n \in \mathbb{N}$ there is a nonempty closed set $W_n \subset V_n$ such that $\mathrm{int}_{W_n}(C_q(f|W_n)) = \emptyset$. Let $W = Y \cup \bigcup_{n=1}^{\infty} W_n$. Then $W \in A_5$ and $\mathrm{int}_W(C_q(f|W)) = \emptyset$ contrary to $f \in H_{05q}$. So $H_{05} = H_{05q}$. The proofs of the inclusions $H_{06} \subset H_{03}$ and $H_{06q} \subset H_{03q}$ are the same as the proof of the inclusion $H_{05} \subset H_{01}$. So $H_{03} = H_{04} = H_{06}$ and $H_{03q} = H_{04q} = H_{06q}$. Since $H_{06} = H_{06q}$, we have $H_{03} = H_{04} = H_{06} = H_{03q} = H_{04q} = H_{06q}$. The proofs of the inclusions $H_{06} \subset H_8$, $H_{06q} \subset H_{8q}$ are similar to the proof of the inclusion $H_{05} \subset H_{01}$.

Let $f \in H_8$ and let $X \in A_6$. Assume that $\mathrm{int}_X(C(f|X)) = \emptyset$. Let $Y \subset X \setminus C(f|X)$ be a countable set such that (1) from the proof of Theorem 1. Then $Y \in A_8$ and there is a point $x \in Y$ at which the restricted function $f|Y$ is continuous. Fix $r > 0$ such that $r < \mathrm{osc}(f|X)(x)/2$. There is an open interval $I \ni x$ such that $|f(t) - f(x)| < r/2$ for each $t \in I \cap Y$. From (1) it follows that

(3) $$|f(t) - f(x)| \le r/2 \text{ for each point } t \in I \cap X.$$

Since $\mathrm{osc}(f|X)(x) > 2r$, there is a point $u \in I \cap X$ such that

$$|f(u) - f(x)| > 3r/4$$

contrary to (3). So $H_8 \subset H_{06}$ and $H_8 = H_{06}$.

For the proof of the inclusion $H_{8q} \subset H_{06q}$ we fix $f \in H_{8q}$ and $X \in A_6$. If $\mathrm{int}_X(C_q(f|X)) = \emptyset$, then there is a countable set $Y \subset X \setminus C_q(f|X)$ such that (1) can be proved from Theorem 1. Since $Y \in A_8$ and $f \in H_{8q}$, there is a point $x \in Y$ at which $f|Y$ is quasicontinuous. From the inclusion $Y \subset X \setminus C_q(f|X)$ and from the fact that $x \in Y$ there is $r > 0$ such that

(4) $$([x - r, x + r] \cap X) \times ([f(x) - 2r, f(x) + 2r]) \cap \{(t, f(t)); t \in C(f|X)\} = \emptyset.$$

Let $I \subset (x - r, x + r)$ be an open interval such that $I \cap Y \neq \emptyset$ and

(5) $$|f(t) - f(x)| < r \text{ for each } t \in I \cap Y.$$

Since $f \in H_{8q} \subset H_{8c} = H_{06c}$ and $X \in A_6$, there is a point $x \in I \cap C(f|X)$ contrary to (1), (4), and (5). So $H_{06q} = H_{8q}$. This implies that $H_{08} \subset H_8 \subset H_{06}$ and $H_{08q} \subset H_{8q} \subset H_{06q}$. The proofs of the inclusions $H_{06} \subset H_{08}$ and $H_{06q} \subset H_{08q}$ are similar to the proof of the inclusion $H_{05} \subset H_{01}$. Thus

$$H_{03} = H_{04} = H_{06} = H_{08} = H_8 = H_{03q} = H_{04q} = H_{06q} = H_{08q} = H_{8q}.$$

The function g from the proof of Theorem 1 belongs to $H_{06} \setminus H_{01}$. This completes the proof.

Theorem 3 *We have*

(a) $H_1 = H_2 \subsetneqq H_3 = H_4 \subsetneqq H_6$,

(b) $H_2 \subsetneqq H_5$,

(c) $H_{1q} = H_{2q} \subsetneqq H_{3q} = H_{4q} \subsetneqq H_{6q} = H_6$,

(d) $H_{2q} \subsetneqq H_{5q}$,

(e) $H_5 \setminus H_{3q} \neq \emptyset$,

(f) $H_6 = H_{6q} = H_{6c}$,

(g) $H_5 \subset H_{5q} \subsetneqq H_{5c}$,

(h) $H_{5c} \subsetneqq H_6$,

(i) $H_1 \subset H_{1q} \subsetneqq H_{1c}$,

(j) $H_3 \subset H_{3q} \subsetneqq H_{3c}$.

PROOF. (a) By Remark 1, $H_1 = H_2 \subset H_3 \subset H_4 \subset H_6$. The function g from the proof of Theorem 1 belongs to $H_3 \setminus H_2$. Let $f \in H_4$ and let $X \in A_3$. If $C(f|X) = \emptyset$, then $X = \bigcup_{n=1}^{\infty} X_n$

(6) $$Y = \bigcup_{n=1}^{\infty} \mathrm{cl}\, X_n.$$

Observe that for $n \in \mathbb{N}$

(7) $$\mathrm{cl}\, X_n \subset \{x \in Y;\ \mathrm{osc}(f|Y)(x) \geq 1/n\}.$$

Since $X \subset Y \subset \mathrm{cl}\, X$, since $X \in A_3$ and since Y is an F_σ set, $Y \in A_4$. Thus there is a point $y \in C(f|Y)$, in a contradiction with (6) and (7). So, $H_3 = H_4$. Let (G_n) be a sequence of nowhere dense sets belonging to A_6 such that $G_i \cap G_j = \emptyset$ for $i \neq j$ $(i, j \in \mathbb{N})$ and $m(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} G_n) = 0$. Put

$$h(x) = \begin{cases} 1/n & \text{for} \quad x \in G_n,\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Since h is a Baire 1 function, $h \in H_6$. If $C_q(h|(\bigcup_{n=1}^{\infty} G_n)) \neq \emptyset$, then there is a point $x \in \bigcup_{n=1}^{\infty} G_n$ at which $h|\bigcup_{n=1}^{\infty} G_n$ is quasicontinuous. Let $x \in G_{n_0}$. There is an open interval I such that

(8) $$h(t) \geq 1/(2n_0) \text{ for each } t \in I \cap \bigcup_{n=1}^{\infty} G_n.$$

Let $y \in I \setminus \cup_{n=1}^{\infty} G_n$. Then $y \in C(h)$ and $h(y) = 0$ contrary to (8). So

$$h \in H_6 \setminus H_{4q} \subset H_6 \setminus H_4.$$

(b) By Remark 1, $H_2 \subset H_5$. The function $h \in H_5 \setminus H_2$.

(c) By Remark 1, $H_{1q} \subset H_{2q} \subset H_{3q} \subset H_{4q} \subset H_{6q}$ and $H_6 \subset H_{6q}$. Let $f \in H_{2q}$ and let $X \in A_1$. If $C_q(f|X) = \emptyset$, then $X = \cup_{n=1}^{\infty} X_n$, where

$$X_n = \{x \in X;\ x \notin \mathrm{cl}(\mathrm{int}_X(\{t \in X;\ |f(t) - f(x)| < 1/n\})),\ n \in \mathbb{N}.$$

If $Y_n = \mathrm{cl}\, X_n$ and $Y = \cup_{n=1}^{\infty} Y_n$, then Y is an F_σ-set and

$$Y_n \subset \{x \in Y;\ x \notin \mathrm{cl}(\mathrm{int}_Y(\{t \in Y;\ |f(t) - f(x)| < 1/n\})).$$

Consequently $Y \in A_2$ and $C_q(f|Y) = \emptyset$ contrary to $f \in H_{2q}$. So $H_{1q} = H_{2q}$. The proof of the inclusion $H_{4q} \subset H_{3q}$ is analogous. So $H_{3q} = H_{4q}$. The function g from the proof of Theorem 1 belongs to $H_{3q} \setminus H_{2q}$. The function h belongs to $H_{6q} \setminus H_{4q}$. Let $f \in H_{6c}$ and let $X \in A_6$. There are open intervals I_n, $n \in \mathbb{N}$, such that for $n \in \mathbb{N}$

$I_n \cap X \neq \emptyset$, $\mathrm{cl}\, I_{n+1} \subset I_n$, the diameter $d(I_n) < 1/n$, and osc $f < 1/n$ on I_n.

Then $\cap_{n=1}^{\infty} I_n$ is a singleton set $\{x\} \subset X$ and $f|X$ is continuous at x. So $H_{6c} \subset H_6$ and consequently $H_{6c} = H_{6q} = H_6$. This finishes the proof of (c) and proves (f).

(d), (e) and (g) The function h belongs to $H_5 \setminus H_{3q} \subset H_{5q} - \setminus H_{2q}$. By Remark 1, $H_{2q} \subset H_{5q}$. Let $F \subset [0, 1]$ be a nowhere dense set belonging to A_6. In each component I_n ($n \in \mathbb{N}$) of the set $\mathbb{R} \setminus F$ we find a Cantor set F_n of measure zero and two non Borel sets $F_{n,1}, F_{n,2}$ such that $F_n = F_{n,1} \cup F_{n,2}$ and $F_{n,1} \cap F_{n,2} = \emptyset$. There are two disjoint sets $\mathbb{N}_1, \mathbb{N}_2$ such that $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ and both the sets $\cup_{n \in \mathbb{N}_1} F_n$ and $\cup_{n \in \mathbb{N}_2} F_n$ are dense in F. Put

$$k(x) = \begin{cases} 1/n & \text{for} \quad x \in F_{n,1},\ n \in N_1 \\ 1 - 1/n & \text{for} \quad x \in F_{n,1},\ n \in N_2 \\ 1 & \text{for} \quad x \in F_{n,2},\ n \in N_2 \\ 0 & \text{otherwise} \end{cases}.$$

Then $k \in H_{5c} \setminus H_{5q}$. The inclusions $H_5 \subset H_{5q} \subset H_{5c}$ follow from Remark 1.

(h) By Theorem 1, $H_{5c} \subsetneq H_{6c} = H_6$.

(i) By Remark 1, $H_1 \subset H_{1q} \subset H_{1c}$. The function k belongs to $H_{1c} \setminus H_{1q}$.

(j) By Remark 1, $H_3 \subset H_{3q} \subset H_{3c}$. The function h belongs to $H_{3c} \setminus H_{3q}$.

Problems. *Are the following equalities true?*

(a_1) $H_1 = H_{1q}$.

(a_2) $H_3 = H_{3q}$.

(a_3) $H_5 = H_{5q}$.

References

[1] W. W. Bledsoe, *Neighbourly functions*, Proc. Amer. Math. Soc. **3** (1972), 114–115.

[2] Z. Grande, *Quelques remarques sur les familles de fonctions de Baire de première classe*, Fund. Math. **84** (1974), 87–91.

[3] Z. Grande, *Sur la mesurabilité des fonctions de deux variables*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **21** (1973), 813–816.

[4] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math. **19** (1932), 184–197.

[5] T. Natkaniec, *On some problem of Grande concerning B_1^* functions*, (in preparation).

[6] T. Neubrunn, *Quasi-continuity*, Real Analysis Exchange **14** (1988–89), 259–306.